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MATHEMATICS
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HIGH SCHOOL MATHEMATICS

FIRST COURSE

TEACHERS' EDITION

UNIT TWO 1957-58

NUMBERS

NUMERALS - PRONUMERALS

UNIVERSITY OF ILLINOIS
COMMITTEE ON
SCHOOL MATHEMATICS

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Unit 2

NUMBERS - NUMERALS - PRONUMERALS

2.01 True or False. --You can't answer questions with holes in them.

2.02 Symbols instead of blanks. --Sammy wonders how you add letters. You have to substitute numerals for the same number in the same-shaped blank.

2.03 Pronouns in mathematics. --'It' has the same purpose in the English language as 'x' has in mathematics.

2.04 Writing expressions for numbers. --A number has many names and we usually want a simple-looking name.

2.05 Pronumerals and principles of arithmetic. --To state a general principle about numbers you need pronumerals.

2.06 Pronumerals and letter symbols. --It is easier to write letters than to draw boxes.

2.07 Pronumerals and directed numbers. --Statements of rules for arithmetic with directed numbers contain pronumerals.

2.08 Algebraic expressions. --Equivalent expressions become names for the same number when you substitute.

UICSM
University High School
Urbana, Illinois
1955

TEACHERS COMMENTARY

Introduction

This unit treats one of the most important ideas in mathematics--the idea of a variable. Since this is the first time the student will receive formal instruction in the use of variables, the instruction needs to be given with considerable care. Too often students come to their study of algebra with misconceptions about variables, and leave their study with many of these misconceptions still intact.

One of the innovations we have made in the teaching of this concept is a change in terminology. In the early years of the UICSM program, we used the term 'general number' to denote a variable. And we noticed that this usage seemed to have unfortunate results. For example, many of our students felt (during a final examination) that the sentence:

$-x$ is a negative number

was a true one. [It is neither true nor false. See T. C. 20A ff.] This led us to believe that our students thought not only that numbers were marks on paper but that letters of the alphabet could be numbers. And, if one goes about calling the letters he uses in equations, inequalities, and other sentences, 'general numbers', it is understandable that the novice will regard these letters as numbers. Alternatives to 'general number' such as 'literal number' and 'unknown number' also carry unfortunate connotations. The word 'variable' itself carries the connotation of change. But in the equation '$x^2 + 5 = 9$' the use of the variable 'x' implies no notion of change.

We went to the mathematical logician in order to clarify for ourselves, and so for our students, the precise role that letters play in mathematical sentences. For one thing, we learned that a variable is a mark on paper. We do not regard a variable as something which is denoted by a mark as, for example, a number is denoted (or, named) by a numeral. A variable is not a fuzzy thing which "jumps all over the place". In the equation '$x^2 + 5 = 9$', the letter 'x' is a variable, and it is not the case that the letter 'x' stands for a variable.

A second thing we learned is that a variable does not have a referent. A variable, although it is a symbol, is not a name of an object. In fact, a variable is nothing more than a blank in an expression. For example, the blank in:

$$_____ + 2 = 9$$

is a variable. It is a symbol which holds a place for a name of an object. In the sentence:

$$_____ + 2 = 9,$$

you can think of the blank as holding a place for a name of, say, a real number. When the blank is replaced by a numeral of a real number, the sentence is converted into either a true statement or a false statement. The only essential difference between the sentences:

$$_____ + 2 = 9 \qquad \text{and:} \qquad x + 2 = 9$$

is that the former uses a blank as a variable, while the latter uses a letter as a variable.

The similarity between the role of variables in mathematical sentences and the role of pronouns in English sentences led us to coin the word 'pronumeral', and to use it instead of 'variable'. [Actually, 'pronumeral' is used instead of 'numerical variable'. A variable can also hold a place for a name of an object other than a number. Later in the UICSM program when variables are used in sentences which, for example, talk about points rather than numbers, we introduce the word 'variable', and note at that time that 'pronumeral' refers to a rather special kind of variable.]

In order to combat the misconceptions students have about the role of letters, we introduce them to variables by using, as pronumerals, such symbols as: $\square$, $\bigcirc$, $\triangle$, $_____$, and $\hexagon$. The transition from frames to letters is gradual.

Thus, there are three things about pronumerals which we stress:

(1) a pronumeral is a symbol,

(2) a pronumeral is not a numeral, and

(3) a pronumeral is a symbol which holds a place for a numeral.

* * *

In response to requests from elementary school teachers and supervisors for suggestions on how to introduce some of the UICSM ideas into the teaching of arithmetic, the staff wrote the article, "Arithmetic With Frames", which appeared in the April 1957 issue of The Arithmetic Teacher. As long as our supply of reprints lasts, we shall be glad to send copies by request.

We open this unit on the use of letters in mathematical sentences with a carefully worked-out "gimmick" designed both to entertain the student and to help him build correct ideas. Although our development is completely different from that found in existing textbooks in high school mathematics, it is in line with the concept of the role of 'x', 'y', etc. most widely accepted by logicians. [We hope you will find time to look into Tarski's Introduction to Logic and compare his treatment of variables with our development.] We do not anticipate that the student will have trouble with this unit; however, you may have some difficulty in eliminating the traditional explanations and terminology from your own thinking and conversation. If you are also teaching traditional classes from a conventional textbook, your problem is even more difficult. We trust that you will bear with us even though it may mean schizophrenia for you.

* * *

The student will need to read the material at the top of page 2-3 in order to understand completely Miss Adam's way of giving True-False tests. The Class Exercise on page 2-3 should clarify the procedure for the student.

* * *

The colored insert behind page 2-2 serves to make the holes stand out more clearly. The student should not make any marks on page 2-2 since it will be referred to again later in the unit.

2.01 True or False.--Miss Adams who teaches mathematics in Zabranchburg Junior High School has an interesting way of preparing True-False tests. First, she mimeographs one page of items for each student in her class. Then she uses a paper punch to make holes in various spots on these pages.

Turn the page to see the first page of Miss Adams' test.

Name.....

Class.....

Date.....

TRUE-FALSE TEST

Instructions: Write 'T' in the space to the left of an item if the statement is true. Write 'F' in this space if the statement is false.

.... 1. $3 + 7 = 10$

.... 2. $2 - 1 = 3$

.... 3. $5 + \square = 8$

.... 4. $4 - (-3) = \square$

.... 5. $8 \times 6 = 45$

.... 6. $3 \times \square = 15$

.... 7. $\square \div 4 = -12$

.... 8. $3 \div 2 = 1.5$

.... 9. $8 \div \square = 10$

.... 10. $\square \times 6 = 24$

.... 11. $8 \times \square = \square$

.... 12. $2 \times 15 = \square$

.... 13. $\frac{1}{2} \times 9 = \square$

.... 14. $8 + \square = -12$

.... 15. $(-2) + \square = -42$

.... 16. $\square \times 5 = \square$

.... 17. $(-2) + (-4) = -8$

.... 18. $7 \div \square = \square$

.... 19. $(-3) \times \square = \square$

.... 20. $\square \times \square = \square$

Instruct the students (if they have not hit upon the idea themselves) on how to make the second page of the test. Tell them to insert an ordinary sheet of blank paper behind page 2-2 and to write numerals, on this inserted sheet, through the holes in the page. Be very certain that the students choose numerals so that about half of the completed statements are false, since there is a tendency to convert all of the sentences into true statements. Check this by walking among them as they compose the second page. [For an explanation of how we use 'sentence' and 'statement' in the Teachers Commentary, see T. C. 20A ff.]

* * *

Exercise 3 of Part D is a fooler. If a student has followed the exercises to this point, he will think that this question is ill-conceived. Naturally, there is no single answer which every student should have for this item. Some students will have composed false items and some will have composed true items. The student should be able to foresee this situation without checking among his colleagues. But, if some of the students do not understand that the answers should differ, you can ask for a show of hands of those who have 'T' and for a show of hands of those who have 'F'.

Next, she mimeographs several different second pages. Each second page can be slid under the first page and has numerals in positions corresponding to the holes in the first page. When a student takes this test, Miss Adams gives him a copy of the first page and a copy (out of several possible copies) of the second page. Then he fastens the pages together and is ready to work on the test. With this system a student is able to take several tests in one class period using the same first page.

CLASS EXERCISE

A. Make up your own second page for the True-False test on page 2-2. Choose your numerals for the second page so that about half of the items on the test are false. Now, take the test and record your answers ('T' or 'F') on another sheet of paper. (Record answers in a single column.)

B. Exchange second pages with your neighbor. Take this new test and record your answers in a column alongside the answers to the first test.

C. Repeat Part B by exchanging the second page you now have for that of another neighbor.

D. Look at your three columns of answers.

1. Is there a 'T' for item 1 in each column? Should every student in the class have written 'T' for item 1?
2. Is there an 'F' for item 2 in each column? Should every student have written 'F' for item 2?
3. What answer should every student have for item 3? Explain.
4. For what items on the test should every student have the same answer?
5. Explain why it is unlikely for every student to have the same answer for, say, item 18?

Such a student should realize that, while he has fulfilled the instructions that he write in each blank a numeral for the same number, in order to know that he has done so he has to know that $3 + 1 = 8 \div 2$. A student who writes '4' in each blank, or who writes '$8 \div 2$' in each blank does not need such additional information. In any case, the number which the student should find is 4, whether he calls it '4', 'IV', '$\frac{12}{3}$', or '57 - 53'.

* * *

These exercises should be handled on a very informal basis. Needless to say, this is not the place to introduce formal notions of equation-solving! You should work several exercises with the students to be sure they know how to proceed. Give the students complete freedom in devising their own methods for discovering correct answers. You may have to caution students against getting help from home on these exercises.

You may want to encourage some discussion about why the students in Miss Adams' class were unable to answer most of the items. Let your students decide which items could be answered without a second page. Elicit from the students the idea that an incomplete statement [i.e., an open sentence] about numbers is neither true nor false. Do not attempt to get a sophisticated explanation from your students at this time.

* * *

In each of the exercises starting at the bottom of page 2-4 and continuing through the first half of page 2-7, the student must find a number which satisfies all of the sentences in the exercise. The student will probably begin by looking for a numeral which when written in all the blanks yields true statements. For example, in the case of the Sample, a student might choose the numeral '4' because he recalls that '7 = 4 + 3' is true. In this case he would write:

4 is a number. If I add 3 to 4, the sum is 7.

Another student might recognize that '7 - 3' is an appropriate numeral, and would write:

7 - 3 is a number. If I add 3 to 7 - 3,
the sum is 7.

Any other numeral for 4, say '8 ÷ 2', or '3 + 1', might be used in filling the blanks in the Sample. A student might even write:

3 + 1 is a number. If I add 3 to 8 ÷ 2,
the sum is 7.

(continued on T. C. 4B)

TRUE OR FALSE ? ? ?!!!

Mr. Edwards, principal of Zabranchburg Junior High, took over Miss Adams' class one day when she was ill. She left instructions for Mr. Edwards to give her class the True-False test she had prepared. But she neglected to tell Mr. Edwards to give out the second page also. Mr. Edwards distributed the first page to the class and told them he would collect papers at the end of ten minutes. In the meantime:

NO TALKING! AND NO QUESTIONS!

Of course, no student had any trouble telling that the first statement was true and the second statement was false. Item three puzzled the students. They wished they could ask Mr. Edwards for the second page but they remembered:

NO TALKING! NO QUESTIONS!

Why were the students puzzled? Why could they answer item 5 but <u>not</u> item 6, for example? Think carefully about <u>why</u> the students were unable to answer most of the items. You will be thinking about a very important mathematical idea.

EXERCISES

Below are several exercises each having blanks. For each exercise find a number such that when numerals for this number are put in <u>all</u> of the blanks in that exercise, the statements in the exercise are all true. In some exercises there may be more than one number that will do the job, in some there may be none.

<u>Sample.</u>

_____ is a number. If I add 3 to _____, the sum is 7.

To save you time, here are answers:

1. 6	2. 1	3. -1
4. -24	5. 20	6. 3

7. The required number is 0. You may find a tendency among some students to use two numbers in order to get true statements. Thus, we have added the caution note in parentheses.

8. -10 and +10
   [You might want to ask the class what answer they would give for Exercise 8 if '100' is interpreted as a numeral of a number of arithmetic rather than as a short name of +100.]

9. +9

Solution. Pick a number at random, say, 9. Write its name in each blank:

__9__ is a number. If I add 3 to __9__, the sum is 7.

The last sentence is false, so try another number. Keep trying until you get true sentences. Try 4:

__4__ is a number. If I add 3 to __4__, the the sum is 7.

With 4, both sentences are true.

1. _____ is a number. If I multiply _____ by 7, the product is 42.

2. _____ is a number. If I subtract -3 from _____, the difference is 4.

3. _____ is a number. If I add 2 to _____, the sum is 1.

4. If I divide the number _____ by -3, the quotient is 8.

5. _____ is an even number. If I divide _____ by 4, the quotient is 5.

6. _____ is a number. If I add _____ to _____, the sum is 6. If I add _____ to 6, the sum is 9.

7. _____ is a number. If I add _____ to _____, the sum is _____. (Remember, each time you try a number, put a numeral for that number in every blank!)

8. _____ is a number. If I multiply _____ by _____, the product is 100. (There are two numbers which will work!)

9. _____ is a positive number. If I multiply _____ by _____, the product is 81.

10. $6\frac{1}{2}$ 11. 16 12. 5

13. As in Exercise 7, you may find that some students hesitate about selecting 0. Use this opportunity to impress upon the students that 0 is a perfectly respectable number.

14. Your students may object to Exercise 14. Some will claim that the exercise is in error since there is no number which will work. Some will suggest that 0 is the required number because "nothing works and 0 is nothing". Point out to students that in this course they should expect to find problems whose solutions are somewhat unconventional. An acceptable answer to this exercise is that there is no number which works.

15. Students should supply all three of the required numbers, 3, 5, and 7.

16. 3, 5, 7, 9 17. +7, -7

18. Every positive number and 0. [Stress here that since 0 is neither positive nor negative, it is not sufficient to answer, simply, "Every positive number".]

19. Every negative number. 20. Every number.

21. Every number 22. 7

10. _____ is a number. If I multiply _____ by 2 and add 4 to that product, the sum is 17.

11. _____ is a number. If I add _____ to 8 and divide that sum by 3, the quotient is 8.

12. If I subtract 5 from the number _____ and multiply the difference by 5, the product is 0.

13. _____ is a number. If I multiply _____ by 4, the product is _____.

14. _____ is a number. If I add _____ to 4, the sum is _____.

15. _____ is an odd number. _____ is greater than 2. _____ is less than 9.

16. _____ is an odd number. _____ is greater than 2. _____ is less than or equal to 9.

17. _____ is a number. The absolute value of _____ is 7.

18. The absolute value of the number _____ is _____.

19. _____ is not 0. The absolute value of _____ is the opposite of _____.

20. The sum of _____ and 2 is equal to the sum of 2 and _____.

21. _____ $\times$ 3 = 3 $\times$ _____.

22. _____ < 10. _____ > 1. _____ is a whole number. _____ is not exactly divisible by 2 nor by 3 nor by 5.

23. 11, 22, 33, 44, 55, 66, 77, 88, 99

24. 0 and 1

25. Every number except 0.

26. No number.

27. Every number.

28. Every number.

* * *

We think that Sammy's attitude is a common one among beginning students. You might ask the students if they also believe as Sammy does that algebra is arithmetic with letters. If one thinks of algebra as arithmetic with letters, it is perfectly reasonable to wonder, "Does $a + b = c$?" or, "Does $x - y = v$?" and to feel that once one learns the correct combinations of letters, one can do algebra just as one did arithmetic.

23. A name of _____ is a two-digit numeral. If the digits in the numeral are reversed, the result is a two-digit numeral which is still a name of _____.

24. _____ × _____ = _____.

25. _____ ≠ _____ + _____.

26. _____ ≠ _____.

27. _____ ≥ _____.

28. _____ ≤ _____.

2.02 Symbols instead of blanks.--Sammy, a student in Miss Adams' class, heard his older brother say that algebra was a lot different from arithmetic. For example, he heard that in algebra you add letters not just numbers. Sammy was puzzled by his brother's remark because he didn't know how to add letters. Since Sammy was hardly an expert in mathematics, he supposed that a few of the "whizzes" in his class might know how to add or multiply letters. But, he didn't want to approach them directly with his problem.

The very next day he thought he had found an opportunity for learning how to do arithmetic with letters. In class Miss Adams gave out the first page of a new True-False test. She told the students that she was not going to hand out second pages this time but, instead, she wanted each student to make up his own second page [as you did on page 2-3] and hand it to his neighbor. In this way, each student made a test for his neighbor. Can you guess the kind of second page Sammy wrote?

Sammy exchanged second pages with Fred. (Fred's nickname was 'The Brain'.) Fred answered the first two items and then stopped in bewilderment. His test looked like this:

Name Fred
Class Math I
Date October 20

TRUE-FALSE TEST

Instructions: Write 'T' in the space to the left of an item if the statement is true. Write 'F' in this space if the statement is false.

.... 1. $4 + 12 = 17$

.... 2. $9 \times 6 = 54$

.... 3. $3 + a = 2$

.... 4. $5 \times b = 25$

.... 5. $8 - 9 = -1$

.... 6. $7 \times x = y$

.... 7. $w + 2 = 6$

.... 8. $3 \times 7 = 34$

.... 9. $8 + y = 10$

.... 10. $m \times 7 = 34$

.... 11. $12 \times x = q$

.... 12. $\frac{2}{3} \times 7 = t$

.... 13. $1.4 + z = -7$

.... 14. $8 + x = p$

.... 15. $(-5) + a = -2$

.... 16. $i \times 7 = g$

.... 17. $k + k = -8$

.... 18. $35 + t = s$

.... 19. $y \times x = z$

.... 20. $x \times 1 = x$

The essence of the concept of variable is contained in Fred's discovery of the similarity between the letters and the holes in the paper. Both the letter and the hole in the paper are marks which hold places for numerals (but not for numbers). Try to get the students to explain this idea.

* * *

Part A is designed to clinch the idea. Since we do not use letters again for a considerable part of the unit, do not spend much time on the similarity between the use of holes and the use of letters. The brief introduction to the parallelism illustrated by these exercises is all we need at this time. [Stress the idea that the circle is supposed to suggest a hole in the paper, and that it is difficult and expensive to print pages with holes in them.]

* * *

In Part B on page 2-10, we introduce different kinds of holes in the paper. Since we shall use these symbols extensively in the subsequent pages, your students will want to give them short names. They may suggest names like 'circ', 'box', and 'hex'. Remind your students, however, that these are names for various shapes of holes in the paper and not names for numbers.

Fred raised his hand and said, "Miss Adams, I don't know how to do this test." Miss Adams was very surprised and so was the rest of the class, for there never seemed to be a problem that Fred couldn't solve. Sammy thought to himself that now he would never find out how to do arithmetic with letters. Fred said that Sammy had put letters instead of numerals on the second page. He said, "It's impossible to tell whether '3 + a = 2' is true or false because . . ." Suddenly, Fred stopped talking. He seemed to be thinking very hard. Then he said,

> "I know. The thing wrong with this test is just what was wrong with the test when Mr. Edwards was here."

Fred was right. Can you tell why?

EXERCISES

A. Each exercise below has two parts. Here is a sample:

(a) $5 + \bigcirc = 12$

(b) $5 + y = 12$

The symbol '$\bigcirc$' in part (a) is a convenient mark which holds a place for you to write a numeral. After you have written the numeral, you have a statement which is either true or false. Part (b) looks like part (a) except that instead of '$\bigcirc$' there is 'y'. Cross out the 'y' and write the same numeral next to the crossed-out 'y' that you wrote in the '$\bigcirc$'. You will then have a statement in part (b) which is either true or false. Here is an example:

(a) $5 + (3) = 12$

(b) $5 + \not{x}\,^{3} = 12$ } False

Students and teachers last year complained of the amount of time and effort required to make the squares, circles, and hexagons required in such exercises as those in Part B. There is no particular benefit for a student in spending the time required to make these frames. Therefore, to expedite homework and homework checking, we are providing you with mimeographed homework sheets which you can distribute to your students. There is ample room provided on these sheets for students to carry out computations as needed. The students will need to use these sheets in conjunction with the textbook since we are not repeating instructions for the exercises. [Homework paper is provided for Part B, pages 2-10 to 2-13.]

* * *

We are also providing you with extra sheets in certain cases so that you may repeat an assignment using different replacements for the pronumerals. Although we hope that you will not have to give additional assignments, we know that some of your students will need more practice in computing, especially with fractional numbers.

Here is another example:

(a) $5 + \bigcirc = 12$ } True

(b) $5 + x = 12$ } True

In each exercise write a numeral in the '$\bigcirc$' of part (a) and replace the letter in part (b) by this numeral. Tell whether the statements in (a) and (b) are true or false. Then, repeat the exercise so that if the resulting statements in both parts were true the first time, you get 'false' the second time, and vice versa.

1. (a) $3 + \bigcirc = 2$
   (b) $3 + k = 2$

2. (a) $\bigcirc - 4 = 17$
   (b) $k - 4 = 17$

3. (a) $\bigcirc + 7 + -3$
   (b) $t + 7 = -3$

4. (a) $7 \times 10 = \bigcirc$
   (b) $7 \times 10 = M$

5. (a) $8 - \bigcirc = 19$
   (b) $8 - x = 19$

6. (a) $(2 \times \bigcirc) + 5 = 11$
   (b) $(2 \times A) + 5 = 11$

7. (a) $7 + \bigcirc > 9$
   (b) $7 + Z > 9$

8. (a) $18 - \bigcirc \leq 3$
   (b) $18 - s \leq 3$

9. (a) $\bigcirc = 3$
   (b) $k = 3$

10. (a) $\bigcirc \neq 5$
    (b) $m \neq 5$

11. (a) $\bigcirc \times 0 = 0$
    (b) $r \times 0 = 0$

12. (a) $\bigcirc \times 0 = 5$
    (b) $d \times 0 = 5$

B. In each of the exercises below there are symbols like '$\bigcirc$' and '$\square$' and '$\hexagon$'. These symbols hold places in which you are to write numerals. After the numerals have been written you have complete sentences which are either true or false.

Sample. In the expression:

$$\square + \square + 3 = \square - 2$$

(a) write '7' in each '$\square$',

(b) write '-5' in each '$\square$', and

(c) write '5' in each '$\square$'.

After making the substitution indicated in each part, tell whether the resulting statement is true or false.

Solution.

(a) We make the substitution:

$$\boxed{7} + \boxed{7} + 3 = \boxed{7} - 2$$

and we ask ourselves, "Is this a true statement?" The expression '7 + 7 + 3' stands for 17. The expression '7 - 2' stands for 5. Since $17 \neq 5$, we say that the statement:

$$\boxed{7} + \boxed{7} + 3 = \boxed{7} - 2$$

is false.

(b) Substitute '-5' for '$\square$':

$$\boxed{-5} + \boxed{-5} + 3 = \boxed{-5} - 2$$

Since $-5 + (-5) + 3 = -7$, and since $-5 - 2 = -7$, we know that the statement:

$$\boxed{-5} + \boxed{-5} + 3 = \boxed{-5} - 2$$

is true.

(c) $$\boxed{5} + \boxed{5} + 3 = \boxed{5} - 2$$

$5 + 5 + 3 = 13$

$5 - 2 = 3$

$13 \neq 3$

The statement is false.

ach

to get students to recognize a sentence whose "replacement instances" are instances of a principle of arithmetic. Another good way to do this is to ask, say in Exercise 6, for a number which will convert the sentence into a false statement.

ach

Note that the statements are checked by simplifying each side of the statement separately. Note also that the frames are kept after the numerals have been written in them. Strictly speaking, it is not necessary to keep the frames. However, by keeping the frames visible, the student is helped in checking his work, and in avoiding confusion when several different kinds of frames are used in the same expression.

* * *

Notice that from Exercise 4 on we modify instructions. Instead of telling the student to write a certain numeral in each frame, we tell him to replace the frame by a numeral. This shift should cause no difficulty. After a "hole" has been "filled" by a numeral, the hole "disappears". This has the same result as replacing a frame by a numeral. The only difference between the two procedures is that in the first case, the outline of the frame remains visible. The student probably gains more understanding at the outset by leaving the frames around the numerals so that he can see the expression he started with. However, once a numeral is written in a frame, the frame can be discarded.

* * *

Ask students whether they can find other numbers which will convert the sentences into true statements. This will provide the class with interesting exploratory work for equation-solving.

* * *

Mrs. Schroeder reported good results when she asked her students to predict the results of replacing the pronumerals in Exercises 2 and 6 by numerals for 0, 1, -1, 2, -2, etc. This is a good way

(continued on T. C. 12B)

In each of the following exercises make the substitution indicated in each part and tell whether the resulting statement is true or false.

1. $(2 \times \square) + 5 = (7 \times \square) - 5$

   (a) Write '3' in each '$\square$'.

   (b) Write '-2' in each '$\square$'.

   (c) Write '2' in each '$\square$'.

2. $(3 \times \hexagon) + (2 \times \hexagon) = 5 \times \hexagon$

   (a) Write '4' in each '$\hexagon$'.

   (b) Write '0' in each '$\hexagon$'.

   (c) Write '-5' in each '$\hexagon$'.

3. $\bigcirc + (3 \times \bigcirc) = 25$

   (a) Write '$5\frac{1}{3}$' in each '$\bigcirc$'.

   (b) Write '$6\frac{1}{4}$' in each '$\bigcirc$'.

   (c) Write '8' in each '$\bigcirc$'.

4. $\square + 8 = 6 - \square$

   (a) '-1' for '$\square$'

   (b) '6' for '$\square$'

   (c) '2' for '$\square$'

5. $(9 \times \hexagon) + 7 = 4 - (2 \times \hexagon)$

   (a) '-2' for '$\hexagon$'

   (b) '0' for '$\hexagon$'

   (c) '1' for '$\hexagon$'

6. $\bigcirc + 2 = 2 + \bigcirc$

   (a) '3' for '$\bigcirc$'

   (b) '$2\frac{1}{2}$' for '$\bigcirc$'

   (c) '-12' for '$\bigcirc$'

7. $(2 \times \square) + 2 = (2 \times \square) - 2$

   (a) '3' for '$\square$'

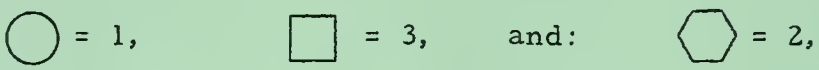
   (b) '-2' for '$\square$'

   (c) '8' for '$\square$'

In Part C be sure the student follows the rule that numerals for the same number are written in the same type of frame in a given expression. [Homework paper is provided for Part C, pages 2-13 to 2-15.]

* * *

Avoid abbreviations like:

 = 1, = 3, and: = 2,

since a frame is not a numeral.

Say, instead:

'1' for '◯', or: '1' replaces '◯', or: '1' in each '◯'.

* * *

In Part C, Exercise 3, the students may recognize that this is the pattern for the associative principle for addition. In so doing, they will discover an important use for pronumerals. Some students may volunteer that any set of replacements will lead to a true statement. Such discoveries should be encouraged. Other principles are suggested by later exercises.

8. $3 \times \bigcirc > (2 \times \bigcirc) + 1$

(a) '3' for '$\bigcirc$'

(b) '0' for '$\bigcirc$'

(c) '-2' for '$\bigcirc$'

9. $|\bigcirc - 2| = 2 - \bigcirc$

(a) '5' for '$\bigcirc$'

(b) '-3' for '$\bigcirc$'

(c) '2' for '$\bigcirc$'

C. The exercises below are similar to the ones in Part B except that the expressions contain more than one of the symbols '$\bigcirc$', '$\square$', and '$\hexagon$'. Make the substitutions indicated in each part and tell whether the resulting statement is true or false.

1. $(2 \times \bigcirc) + \square = 7 - \square + \bigcirc$

(a) Write '1' in each '$\bigcirc$' and write '3' in each '$\square$'.

(b) Write '3' in each '$\bigcirc$' and write '4' in each '$\square$'.

(c) Write '-5' in each '$\bigcirc$' and write '6' in each '$\square$'.

2. $(4 \times \square) - (2 \times \hexagon) = \square + (5 \times \hexagon)$

(a) Write '-7' in each '$\square$' and write '-3' in each '$\hexagon$'.

(b) Write '5' in each '$\square$' and write '2' in each '$\hexagon$'.

(c) Write '0' in each '$\square$' and '0' in each '$\hexagon$'.

3. $(\square + \bigcirc) + \hexagon = \square + (\bigcirc + \hexagon)$

(a) '5' in each '$\square$', '3' in each '$\bigcirc$', '4' in each '$\hexagon$'

(b) '-4' in each '$\square$', '2' in each '$\bigcirc$', '-6' in each '$\hexagon$'

(c) '9' in each '$\square$', '9' in each '$\bigcirc$', '9' in each '$\hexagon$'

When you discuss Exercise 6, some students may recognize that this is the form of the "Pythagorean rule". Ask whether all numbers which "work" for this sentence would give the sides of a right triangle. Take the time necessary to state the Pythagorean rule and its converse, and illustrate them for the class.

* * *

In Exercise 8, ask what replacements must be made in order to get a true statement.

4. $5 \times \bigcirc \times \hexagon = \square$

(a) '8' for '$\bigcirc$', '2' for '$\hexagon$', '20' for '$\square$'

(b) '-1' for '$\bigcirc$', '-2' for '$\hexagon$', '10' for '$\square$'

(c) '0' for '$\bigcirc$', '0' for '$\hexagon$', '0' for '$\square$'

5. $\bigcirc \times \square = \square \times \bigcirc$

(a) '5' for '$\bigcirc$', '$3\frac{1}{5}$' for '$\square$'

(b) '2.4' for '$\bigcirc$', '-2.4' for '$\square$'

(c) '7' for '$\bigcirc$', '$\frac{1}{7}$' for '$\square$'

6. $(\bigcirc \times \bigcirc) + (\hexagon \times \hexagon) = (\square \times \square)$

(a) '3' for '$\bigcirc$', '4' for '$\hexagon$', '5' for '$\square$'

(b) '-7' for '$\bigcirc$', '24' for '$\hexagon$', '25' for '$\square$'

(c) '1.4' for '$\bigcirc$', '1.3' for '$\hexagon$', '5.0' for '$\square$'

7. $(\square + \hexagon) \times \bigcirc = (\square \times \bigcirc) + (\hexagon \times \bigcirc)$

(a) '2' for '$\square$', '-3' for '$\hexagon$', '5' for '$\bigcirc$'

(b) '$1\frac{1}{2}$' for '$\square$', '$-5\frac{1}{2}$' for '$\hexagon$', '$-8\frac{1}{4}$' for '$\bigcirc$'

(c) '-2.7' for '$\square$', '+4.9' for '$\hexagon$', '-7.3' for '$\bigcirc$'

8. $\bigcirc + \square - \hexagon = \square + \bigcirc + \hexagon$

(a) '3' for '$\bigcirc$', '2' for '$\square$', '0' for '$\hexagon$'

(b) '4' for '$\bigcirc$', '3' for '$\square$', '4' for '$\hexagon$'

(c) '$-3\frac{1}{2}$' for '$\bigcirc$', '$4\frac{1}{2}$' for '$\square$', '0' for '$\hexagon$'

9. $(\square - \bigcirc) \times (\square + \bigcirc) = (\square \times \square) - (\bigcirc \times \bigcirc)$

(a) '9' for '$\square$', '1' for '$\bigcirc$'

(b) '-12' for '$\square$', '12' for '$\bigcirc$'

(c) '$3\frac{1}{2}$' for '$\square$', '$-4\frac{1}{2}$' for '$\bigcirc$'

In discussing Exercise 10, ask the students whether there are any replacements for the pronumerals which will give a true statement.

* * *

In discussing Exercise 11, you may find it profitable to demonstrate for the class that this is a short-cut for squaring certain numbers. Mrs. Catlow reports that her students derived from this exercise their own short-cuts for squaring a number "ending in 5" or "ending in $\frac{1}{2}$".

* * *

After the students have decided which of the three parts of Exercise 12 gives a true statement when the pronumerals are replaced by the numerals, you might ask, as a real challenge, whether they can find any other replacements for the pronumerals which will make a true statement. Help them by suggesting that they select arbitrarily a replacement for ' ◯ ', and then search for a replacement for ' □ '.

* * *

Part D is strictly a set of <u>exploratory</u> exercises. Do not expect all students to be able to write the simplest expression in each case. The important thing to be gained from these exercises is that a student learns to check his simpler but equivalent expression by replacing the pronumerals by numerals. He also learns to eliminate expressions which are not equivalent to the given ones by finding counter-examples. We are very interested in the success of your students with this kind of exercise. [Homework paper is provided for Part D, pages 2-15 to 2-17.]

10. $\bigcirc - \bigcirc = \bigcirc - \bigcirc$

(a) '10' for '$\bigcirc$', '3' for '$\bigcirc$'

(b) '125' for '$\bigcirc$', '-125' for '$\bigcirc$'

(c) '$17\frac{1}{2}$' for '$\bigcirc$', '$-17\frac{1}{2}$' for '$\bigcirc$'

11. $(\bigcirc + \frac{1}{2}) \times (\bigcirc + \frac{1}{2}) = [\bigcirc \times (\bigcirc + 1)] + \frac{1}{4}$

(a) '9' for '$\bigcirc$'

(b) '7' for '$\bigcirc$'

(c) '19' for '$\bigcirc$'

12. $(7 \times \bigcirc) + (2 \times \square) = 9 \times \bigcirc \times \square$

(a) '1' for '$\bigcirc$', '1' for '$\square$'

(b) '3' for '$\bigcirc$', '2' for '$\square$'

(c) '-2' for '$\bigcirc$', '-5' for '$\square$'

D. Suppose you are given the expression:

$$\bigcirc + \bigcirc + \bigcirc$$

Can you find a second expression containing '$\bigcirc$' such that if you pick a number and write a numeral for that number in each '$\bigcirc$' of both expressions, each expression will give a name for the same number? If you experiment for a while, you will probably find that this can be done. One solution is the expression '$3 \times \bigcirc$'. If we write:

$$\bigcirc + \bigcirc + \bigcirc = 3 \times \bigcirc$$

and write a numeral for the same number in each '$\bigcirc$', we get a name for the same number on both sides of ' = '. Try 7. Is the statement:

$$(7) + (7) + (7) = 3 \times (7)$$

a true statement? Do you think you will get a true statement no matter what number you try?

There are other solutions to this problem:

(2 × ◯) + ◯

(4 × ◯) - ◯

(10 × ◯) - (7 × ◯)

Check each of these expressions in the same way as we checked '3 × ◯'.

Now, out of the four expressions we have given, which do you think is the simplest-looking one? Most people would agree that the expression '3 × ◯' is the simplest.

In each of the following exercises you are to write as simple an expression as you can. This expression will contain symbols like '◯' and '□' and '⬡'. In each case, the expression you write must be such that when numerals are substituted properly for each '◯', '□', and '⬡' both in the given expression, and in your simpler expression, you get two names for the same number. This means that when you write your simpler expression at the right of ' = ' and write the given expression at the left of ' = ', you should get true statements every time you make a correct substitution.

(continued on next page)

with the instructions for Part D. They claimed that only an expression for 0 would work. It is permissible for students to write '0' as the required expression; numerals are later classified as algebraic expressions [See page 2-65]. However, if the student insists upon a pronumeral expression, he may write:

$$0 \times \square \ .$$

* * *

The exercises of Part E will be a little more difficult. The student is asked to write a simpler-looking expression which is not equivalent to the given expression, and to find a number which satisfies the resulting equation as well as a number which does not. The purpose here [i.e., in Parts D, E, and F] is to show that there are three kinds of equations: identities, conditionals, and equations which have no roots. The students, of course, should not be told this purpose. [Homework paper is provided for Part E.]

As we said on T. C. 15A, these exercises are exploratory. However, if a student says he cannot find an expression that will work for an exercise such as 3, you may put more examples

[(3 × □) + (3 × □) = , (100 × △) + (100 × △) = , and

then: (8 × ▽) + (11 × ▽) =] of similar expressions on the

board until he recognizes the principle involved. This will be more helpful to the student than to let some other student tell him how to write the simpler expression asked for by the directions. The most important thing in doing all of these exercises, is not that the student write the simplest-looking expression, but that he get an equivalent expression. As you discuss these exercises, stop from time to time and ask the students to justify the expression they have written by reference to the principles involved. For example, consider the expression '10 × ⬡ × ◯ ' [Exercise 10]. The student should be able to back up his claim that, for each pair of replacements for ' ⬡ ' and ' ◯ ' in '5 × ⬡ × 2 × ◯ ' and in ' 10 × ⬡ × ◯ ', you get two numerals for the same number. The justification resides in the fact that each pair of replacements for the frames in the given expression will yield a numeral which can be transformed, by the associative and commutative principles for multiplication, and the arithmetic fact that 5 × 2 = 10, into the numeral obtained by the same pair of replacements for the frames in '10 × ⬡ × ◯ '.

* * *

Exercise 8 of Part D may cause a bit of difficulty. Some students in Miss McCoy's class claimed that they couldn't put a pronumeral expression on the right of ' = ' and obtain an equation in accordance

(continued on T. C. 17B)

1. $\square + \square =$ 2. $\hexagon + \hexagon =$

3. $(2 \times \bigcirc) + (3 \times \bigcirc) =$ 4. $(5 \times \square) + \square =$

5. $(3 \times \hexagon) - (2 \times \hexagon) =$ 6. $\bigcirc + \bigcirc + \square =$

7. $\bigcirc + \square + \bigcirc + \square =$ 8. $\square - \square =$

9. $\dfrac{3 \times \hexagon}{3} =$ 10. $5 \times \hexagon \times 2 \times \bigcirc =$

11. $(\frac{1}{2} \times \bigcirc) + (3\frac{1}{2} \times \bigcirc) =$

12. $(4.87 \times \bigcirc) + \bigcirc =$

13. $4 \times \bigcirc \times 3 \times \square \times 5 \times \hexagon =$

E. The exercises below are just like the ones in Part D except that in this part you are to write a simpler-looking expression which "works" only for some numbers instead of for all numbers.

Sample. $\bigcirc + \bigcirc =$

Solution. If you write '$\bigcirc + \bigcirc = 2 \times \bigcirc$', it will "work" for every number as in Part D. But, if you write, say '3', or '$\bigcirc$', or '$\hexagon$' on the right of ' = ', you would have an expression which "works" for some numbers and does not "work" for others.

1. $\square + \square + \square =$ 2. $(3 \times \bigcirc) + (2 \times \bigcirc) =$

3. $8 - \hexagon =$ 4. $2 \times (\bigcirc + \square) =$

5. $\square =$ 6. $7 =$

d

[Homework paper is provided for Part F.]

* * *

In Part G, encourage the student to be free in his choice of names to replace the frames. He need observe only two restrictions for his choices: the names substituted are names of students in the class, and a name for the same student goes into the same type of frame in each sentence. [The variables in these sentences are not pronumerals. We could call these variables 'pro-student-names'.]

* * *

One of our teachers felt that the exercises in Part G do not contribute much to the students' understanding of variables. What is your reaction?

F. The exercises below are just like the ones in Part D and E except that in this Part you are to write expressions that do not "work" for any numbers at all.

Sample. ◯ =

Solution. If you write '◯ = ◯', this "works" for all numbers; if you write '◯ = 5', this works for 5; if you write '◯ = ⬡', this works when you write a name for the same number in both '◯' and '⬡'.

Are there expressions which do not work for any number?

Actually, there is no limit to the number of solutions to this problem. Here are just two of them:

◯ = ◯ + 1

◯ = ◯ + 8

1. ⬡ =

2. ◯ + ◯ =

3. ◯ + □ =

4. (2 × ◯) + (3 × ◯) =

G. In the expressions which follow you will find symbols like '( )' and '[ ]'. You are to write in place of these symbols the names of students in your class today. After you have made the substitutions, some of the statements will be true and some will be false. Make substitutions and tell whether the resulting statement is true or false.

1. ( ) is a boy and ( ) sits in the front row.
2. ( ) is a girl and ( ) has a brother in this school.

(continued on next page)

The student should react to Exercise 9 in just the same way as he would react to the request to solve the equation

'$\square = \square + 1$'. In the latter case, there are no numbers which satisfy the equation. In Exercise 9, there are no students who satisfy the sentence. You must avoid saying that the sentence in Exercise 9 is false. What is written in Exercise 9 is neither true nor false--it is an open sentence [as are all the sentences in this group of exercises]. If a student says that the sentence in Exercise 9 is false, he should be corrected by asking him if he really meant to say, "You get a false sentence each time you replace the frames by a name of a student." Just as the student would not claim that the sentence in Exercise 7 [or any of the other exercises] is either true or false, he should not claim that the sentence in Exercise 9 is either true or false.

* * *

In Exercise 14 it is impossible to make a true statement because the teacher is not a member of the class. The student may need to read the instructions again. (This is just a slight hint at the idea of the domain of a variable.)

* * *

In Section 2.03 we introduce the word <u>pronumeral.</u> This section needs to be handled with care. We hope that the students have been prepared for the section by the work they did in the preceding pages. Make clear to the student that the sentence:

He was a president of the United States

is neither true nor false. The student must <u>not</u> assume that because he cannot say that the sentence is true, then it must be the case that the sentence is false.

3. ( ) and [ ] live in the same block.
4. [ ] likes mathematics!
5. ( ) hates mathematics!
6. [ ] and ( ) are sisters.
7. [ ] is the tallest student in the class.
8. [ ] and ( ) wear glasses.
9. [ ] is a year older than [ ].
10. [ ] is taller than ( ).
11. ( ) is taller than ( ).
12. [ ] attends the same classes as [ ].
13. ( ) and [ ] are girls and ( ) has longer hair and she is taller than [ ].

    Note: If you replace 'she' by '( )', is the exercise changed? If you replace 'she' by '[ ]', is the exercise changed?
14. ( ) is the teacher of this class.

2.03 Pronouns in mathematics.--Suppose you were an expert on presidents of the United States and could answer just about every important question concerning them. Someone challenges you to tell whether the following expression is true or false:

He was a president of the United States.

The problem seems to be a very simple one for an expert, but, as you can see, it is not possible to answer one way or the other. If you were told the name of some man and you put this man's name in place of 'He' in the given expression, then it would be easy for you to decide whether the resulting statement is true or false.

to find this word in a dictionary. [Nor will their parents!] Mention to the class that in later UICSM courses, the word 'variable' will be used, but that we shall not do so here. Also mention that words like 'unknown', 'literal number', 'general number', are in use in other high school mathematics courses.

* * *

Is it the case with your students that they are really ignorant of the meaning of 'pronoun'? We have heard that students learn so little of grammar "these days" that it is risky to depend upon their formal knowledge of language in teaching them something about language in mathematics.

* * *

[Homework paper is provided for Part A, pages 2-21 to 2-22.]

The last sentence on page 2-20 and the first sentence on page 2-21 may need amplification. English and mathematical expressions can be converted (by replacement of the pronouns) into statements (true or false sentences) if and only if the expressions are sentences. There are other expressions which contain pronouns but which cannot be converted into statements. For example, the English expression 'his father' is converted into a name of Abraham Lincoln's father when 'Abraham Lincoln' is used to replace the pronoun 'his' [and 'his' is thought of as 'he's']. Similarly, the mathematical expression '3 + $\square$ ' is converted into a name of 8 when '5' is used to replace ' $\square$ '.

[Expressions such as 'his father' and '3 + $\square$ ' which can be converted into names of things are, by logicians, called terms. This is a broader use of the word 'term' than is customary in elementary algebra. And, in fact, in the students' text we use it in the customary narrower sense. In the next revision we may introduce 'term' with this broader meaning. The two sentences on pages 2-20 and 2-21 are hinting at this distinction between terms and sentences.]

* * *

It should be very easy for you to get the students to suggest the word 'pronumeral'. Lead them to recognize that

a noun is a name, and that
a pronoun holds a place for a noun;
a numeral is a name, and
a ? holds a place for a numeral.

Let us know of your success here. It is best to tell students that we have coined the word 'pronumeral', for they will not be able

(continued on T. C. 20C)

Let the student replace the word 'He' in the given sentence by a name. The student should understand that the word 'He' does not stand for a person. If the word 'He' stood for a person, that person's name would be 'He'. Rather, the word 'He' stands in place of a name of a person. [Also, when one replaces 'He', he does not put a person in place of 'He'; he puts a person's name in place of 'He'. Similarly, one does not put a number in place of a pronumeral; one puts a number's name there.]

* * *

Note that (1), (2), (3), and (4) are called 'expressions'. The word 'expression' is a catch-all for us. We use 'expression' when referring to a collection of symbols. In our next revision of the students' materials we shall probably call expressions such as (1), (2), (3), and (4), sentences. Sentences such as these are neither true nor false. When the pronumerals in these sentences are replaced by numerals, the sentences are converted into new sentences which are either true or false. In the present student materials, we call these new sentences statements. In order to avoid the feeling of vagueness which accompanies the word 'expression', we suggest that you use the word 'sentence' in these cases, and talk about "sentences being converted into statements". [Of course, statements are sentences, also. But, it is easier to indicate the distinction between sentences which are neither true nor false, and sentences which are either true or false, by using 'sentence' or 'statement', respectively.]

* * *

(continued on T. C. 20B)

You will recall from your study of grammar that the word 'He' is a pronoun and that a pronoun stands in place of a name (or a noun). So, in the expression:

_____ was a president of the United States

you must "fill in the blank" or replace the pronoun (that is, the blank) by a name before you can answer 'true' or 'false'.

In mathematics we use "pronouns" also. Of course, these pronouns are not the ones you use in English. Here are some examples of mathematical expressions which contain pronouns:

(1) _____ is an even number

(2) 7 + ⬡ = 15

(3) □ + ○ = 3 - □

(4) ___ + □ = □ + ___

Expression (1) contains the pronoun '_____'; expression (2) contains the pronoun '⬡'; expression (3) contains the two different pronouns '□' and '○', and expression (4) contains the two different pronouns '___' and '□'. Although the most commonly used symbols for pronouns in mathematics are letter symbols, we are going to use '□', '○', '⬡', and '_____' for some time before we use letters.

Pronouns in English stand in place of nouns, that is, in place of names of things and of people. The pronouns in mathematics stand in place of names of numbers. When you replace a pronoun in an English expression by a name of something or somebody, you usually get a statement which is either true or false.

When you replace a pronoun in a mathematical expression by a name of a number, you sometimes get a statement which is either true or false. Since pronouns in mathematics stand in place of numerals, that is, in place of names of numbers, we shall call the symbols which act as pronouns, pronumerals.

> We shall call symbols such as ' ', ' ', ' ', and '_____' which hold places for names of numbers or any other symbols which hold places for names of numbers,
>
> pronumerals

Remember: A pronoun in English is not a noun (a name of a thing or a person). A pronumeral is not a numeral (a name of a number).

EXERCISES

A. There are expressions which contain pronumerals and which do not give true or false statements when numerals are substituted for the pronumerals. For example, consider the expression:

$$(3 \times \square) + (2 \times \bigcirc)$$

If, say '5' is written for '$\square$' and '4' is written for '$\bigcirc$', we have:

$$(3 \times 5) + (2 \times 4)$$

which is a name for a number. A simpler-looking name for the same number is '23'. When you see the expression:

$$(3 \times \square) + (2 \times \bigcirc)$$

you are not looking at a name for a number. But when the pronumerals are replaced by numerals, you do have a name for a number.

Each of the following expressions contains pronumerals. Replace the pronumerals by numerals as follows:

'◯' by '7' '⬡' by '0'

'□' by '2' '____' by '-3'

Then give a simpler-looking name for the same number.

1. ◯ + □ - ⬡

2. (3 × ◯) + (2 × □)

3. $\dfrac{\bigcirc + \hexagon}{\bigcirc - \hexagon}$

4. $\dfrac{\bigcirc + \underline{\quad} + \hexagon}{\square}$

5. (◯ + ⬡) × (◯ - ⬡)

6. (◯ + □) × (◯ - □)

7. $\dfrac{\bigcirc}{\square} \times \dfrac{\bigcirc}{\underline{\quad}}$

8. $\dfrac{\underline{\quad}}{\bigcirc} \times \bigcirc$

9. (2 × ◯) + (3 × □) + (4.2 × ⬡)

10. 2 × [◯ + (2 × □) + ____]

11. (◯ × □) + (◯ × ⬡) + (⬡ × □)

12. [____ × (◯ + □)] + (◯ × □)

13. ◯ + (____ ÷ □) + (4 × ____)

14. $\dfrac{\bigcirc}{\underline{\quad}} + \dfrac{\hexagon}{\bigcirc}$

15. $\dfrac{2 \times \bigcirc}{3 \times \square} \times \dfrac{\square}{\bigcirc \times \underline{\quad}}$

In discussing the Sample, it is a good idea to ask the students how the simpler expression '(265 × □)' is obtained from (c). This is an opportunity to review the commutative and associative principles for multiplication. The review will be helpful to a student in writing the simpler expressions he needs as answers to the questions posed by the exercises.

* * *

The exercises in Part B are analogous to those in Part D on page 2-72. In a sense they are preparation for those exercises. When a student writes a pronumeral expression as an "answer" to one of these questions, say question 1(b), he is really giving an abbreviation of:

> If one pencil costs ◯ cents, then the cost of 7 pencils is 7 × ◯ cents.

This sentence will be converted into a true statement when each frame in it is replaced by a numeral for a number of cents. The student needs to develop skill in constructing pronumeral expressions, but he must also understand this matter of converting sentences into true statements.

* * *

When students write pronumeral expressions as answers to the exercises in Part B, you should insist that they check their answers by replacing the pronumerals by numerals. <u>They should make this kind of check whenever they are in doubt.</u>

B. Answer the questions in the following exercises.

Sample. If eggs cost 53 cents a dozen, what is the cost of (a) 3 dozen eggs? (b) $\square$ dozen eggs? (c) (5 × $\square$) dozen eggs?

Solution. (a) $3 \times 53 = 159$; \$1.59

(b) $\square \times 53$; ($\square \times 53$) cents

Note that '$\square \times 53$' is not a name for a number but when a numeral is substituted for '$\square$', you get a name for a number. This number is the number of cents in the cost of the eggs.

(c) [(5 × $\square$) × 53] cents

This expression can be simplified by multiplying 5 by 53. A simpler expression for the answer is:

(265 × $\square$) cents

Note that when a numeral for any number is substituted for '$\square$', the expressions '(5 × $\square$) × 53' and '265 × $\square$ ' give two names for the same number.

1. What is the cost of 7 pencils if one pencil costs (a) 2 cents? (b) ◯ cents? (c) (3 × ◯) cents?

2. What is the number of inches in the perimeter of a square if the length of one side is (a) 5 inches? (b) ⬡ inches? (c) (5 × ⬡) inches?

hick,

(d) $1.2 \times$ inches long?

The area of a square whose side is $1.20 \times$ inches long is how many times the area of a square whose side is $.80 \times$ inches long?

3. What is the area of a triangle if (a) its base is 8 inches long and its altitude is 3 inches long? (b) its base is inches long and its altitude is 4 inches long? (c) its base is inches long and its altitude is $\frac{\bigcirc}{3}$ inches long?

The area of a triangle whose base is inches long and altitude is 4 inches long is how many times the area of a triangle whose base is 2 inches long and whose altitude is $\frac{\bigcirc}{4}$ inches long?

4. What is the perimeter of a rectangle whose dimensions are (a) inches by inches? (b) $(3 \times$ $)$ inches by $(2 \times$ $)$ inches? (c) $\frac{\square}{4}$ inches by $(2 \times$ $)$ inches?

The rectangle whose dimensions are $\frac{\bigcirc}{4}$ inches by $(2 \times$ $)$ inches has a perimeter which is how many times that of a rectangle whose dimensions are inches by $(3\frac{1}{2} \times$ $)$ inches?

[Exercises 3 and 4 are hard but should prove useful in giving students an opportunity to extend themselves in operating with pronumeral expressions.]

hick,

It may be necessary to give the students more computational examples before they can find pronumeral expressions. For example, in Exercise 9 on page 2-24, we suggest that you ask students:

> How long will it take a car traveling at a steady rate to travel 30 miles if it travels 1 mile in $1\frac{1}{2}$ minutes? 2 minutes? 3 minutes? 5 minutes?

Then, list the answers as follows:

$30 \times \boxed{1\frac{1}{2}}$ $\qquad$ $30 \times \boxed{2}$

$30 \times \boxed{3}$ $\qquad$ $30 \times \boxed{5}$.

The student needs to become accustomed to working with expressions such as these in order to be able to set up equations in connection with worded problems.

* * *

Here are supplementary exercises for Part B.

1. What is the area of a circle whose radius is (a) 3 inches long? (b) $\square$ inches long? (c) $2 \times \square$ inches long?

   The area of the circle whose radius is $2 \times \square$ inches long is how many times the area of a circle whose radius is $\square$ inches long?

2. What is the area of a square whose side is (a) 4 inches long? (b) $\bigcirc$ inches long? (c) $.80 \times \bigcirc$ inches long?

(continued on T. C. 24B)

3. If there are 100 sheets of paper in a pile 1 inch thick, what is the number of sheets of paper in a pile (a) 9 inches thick? (b) $\square$ inches thick? (c) $(10 \times \square)$ inches thick?

4. What is the area of a square whose side is (a) $4\frac{1}{2}$ inches long? (b) $\bigcirc$ inches long? (c) $(3 \times \bigcirc)$ inches long?

5. What is the area of a rectangle whose dimensions are (a) 5 inches by 12 inches? (b) $\bigcirc$ inches by $\hexagon$ inches? (c) $(4 \times \hexagon)$ inches by $(2 \times \bigcirc)$ inches?

6. What is the perimeter of an equilateral triangle if each of the three equal sides is (a) $17\frac{1}{3}$ inches long? (b) $\square$ inches long? (c) $(5\frac{2}{3} \times \square)$ inches long?

7. What is the length of a side of a square if its perimeter is (a) 72 inches? (b) $\hexagon$ inches? (c) $(8 \times \bigcirc)$ inches?

8. What is the perimeter of a rectangle whose dimensions are (a) 13 inches by 17 inches? (b) $\bigcirc$ inches by $\hexagon$ inches? (c) $(2 \times \bigcirc)$ inches by $(3 \times \bigcirc)$ inches?

9. How long will it take a car traveling at a steady rate to travel 30 miles if it travels 1 mile in (a) $1\frac{1}{2}$ minutes? (b) $\square$ minutes? (c) $(3 \times \bigcirc)$ minutes?

10. If a man sells a certain number of vacuum cleaners in 3 days, then, on the average, how many days will it take him to sell (a) $\frac{1}{3}$ of this number of vacuum cleaners? (b) $\frac{1}{\frac{3}{\square}}$ of this number of vacuum cleaners? (c) $\frac{3}{7 \times \square}$ of this number of vacuum cleaners?

2.04 Writing expressions for numbers.--Consider the expression:

$$3 \times 4 + 2$$

This expression is a name for a number. Suppose you wanted to find another name for that number. It seems that there are at least two things you could do to find another name.

(I) Multiply 3 by 4 and add 2:

$$3 \times 4 + 2 = 12 + 2 = 14$$

(II) Add 4 and 2 and multiply by 3:

$$3 \times 4 + 2 = 3 \times 6 = 18$$

Now, procedure (I) tells you that another name for the number $3 \times 4 + 2$ is '14'; procedure (II) tells you that another name for the number $3 \times 4 + 2$ is '18'. Obviously, at least one of these two procedures must be incorrect because '14' and '18' are names for different numbers.

One way to clear up this difficulty is to punctuate the expression '$3 \times 4 + 2$' with parentheses. Thus, you can write:

$$(3 \times 4) + 2$$

when you mean 14; you can write:

$$3 \times (4 + 2)$$

when you mean 18.

Now, parentheses tend to make expressions look complicated. So, mathematicians have agreed that if in certain cases they do not write parentheses, the resulting expression should be interpreted in a particular way. For example, in an expression like '$3 \times 4 + 2$' they have agreed that "multiplication is to be carried out before addition". On the basis of the agreement, we can write:

In the discussion on the order of operations to use in simplifying expressions, no illustration has been given for the case where the numerator and denominator of a fraction are expressions in which more than one operation is indicated. This situation may cause difficulty for the students when they try to do some of the exercises on page 2-29. Hence, you may want to take some time to consider expressions such as:

$$\frac{(-4)(3) + 5(-2)}{(3)(-11) - 4(-5.5)} .$$

In doing so, it may be helpful first to consider another expression for the same number:

$$[(-4)(3) + 5(-2)] \div [(3)(-11) - 4(-5.5)] .$$

If they recognize that the expression to the left of '÷' names a number, and the expression to the right of '÷' also names a number, they may then understand that the statement in the box on page 2-26 may be followed in getting a simpler name in each part of the fraction before the final operation of division is performed.

'3 × 4 + 2' instead of '(3 × 4) + 2'

Similarly by agreement, we can write:

'2 + 3 × 4' instead of '2 + (3 × 4)'

and:

'5 × 3 + 4 × 6' instead of '(5 × 3) + (4 × 6)'

Study the following statements. They illustrate some of the agreements mathematicians have reached in writing expressions. Note that unless an agreement had been reached telling "which operations should be carried out first", each of the expressions could be interpreted in at least two different ways.

'5 × 9 - 3'	instead of	'(5 × 9) - 3'
'6 ÷ 2 + 1'	instead of	'(6 ÷ 2) + 1'
'3 ÷ 6 - 3'	instead of	'(3 ÷ 6) - 3'
'5 × 3 - 2'	instead of	'(5 × 3) - 2'
'4 + 6 ÷ 3'	instead of	'4 + (6 ÷ 3)'
'9 - 6 ÷ 2'	instead of	'9 - (6 ÷ 2)'
'8 - 5 × 1'	instead of	'8 - (5 × 1)'
'3 × 5 - 2 × 4'	instead of	'(3 × 5) - (2 × 4)'

Note that the agreements illustrated above are applications of the more general agreement:

> Multiply and divide before you add and subtract.

Of course, the only reason for this agreement is to make it possible to omit parentheses in certain expressions and not cause confusion. Whenever you have any doubt about the interpretation of an expression you have written, you should use parentheses and other grouping symbols to eliminate the doubt.

Miss Blair suggests that we tell students at this point that when numerals replace the pronumerals in an expression such as '$\square\,\bigcirc$' either a multiplication sign of some sort should be inserted between the numerals, or else the numerals should be enclosed in parentheses. Students will find their first need for this convention on page 2-30 in Exercise 5 of Part C.

We suggest that, at this point, you avoid such usages as:

multiply □ by ○ ,

and: find the product of □ and ○ .

Technically speaking, the only things which can be multiplied are numbers, and products of numbers are themselves numbers which can be expressed by writing numerals. One does not multiply a pronumeral by a pronumeral [or, a numeral by a numeral, either]. When one knows what he is talking about and says:

(1) If you multiply □ by ○ , you get □○ ,

what he means is that each replacement instance of sentence (1) is a true statement. He could also mean that a formula for the product of a number and a number is: □○.

We have found that even when textbook and teacher sedulously avoid colloquialisms such as: Multiply '□' by '○', students will invent such colloquialisms for themselves in order to avoid the correct but longer verbal descriptions. Therefore, instead of trying to fight this tendency, simply explain the colloquialism the first time a student uses it in class. Every once in a while after that, ask a student to explain the usage. [Similar comments hold for such colloquial expressions as: add '□' and '△', find the product of '3x' and '2y', and: add '8x' to both sides of the equation.]

* * *

(continued on T. C. 27B)

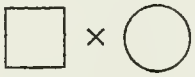
WAYS OF NAMING PRODUCTS

The following five expressions each stand for the same product:

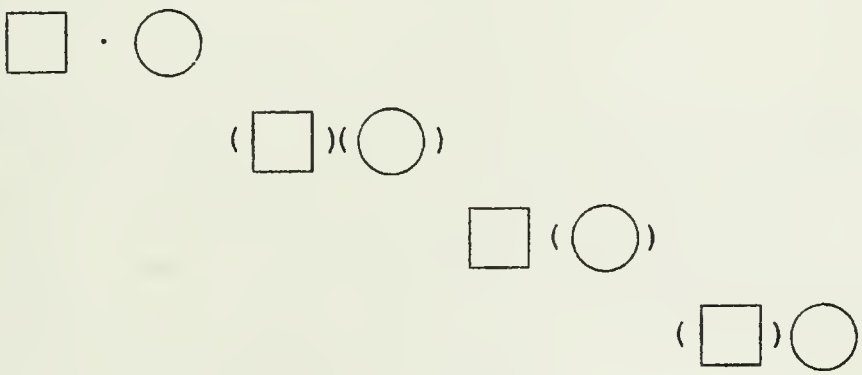
3×4 $3 \cdot 4$ $(3)(4)$ $3(4)$ $(3)4$

You can write a raised period, ' · ', between two numerals whenever you can write a times sign, ' × ', between them. Instead of writing ' × ' or ' · ', you can enclose one or both of the numerals in parentheses and not write any sign between the numerals so enclosed.

In writing pronumeral expressions, we follow corresponding procedures. Instead of writing:

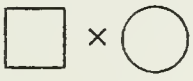


You can write any of the following:



It is customary to write merely:

instead of:



SIMPLER EXPRESSIONS

You have learned that some expressions are names for numbers. For example, the expression '$3 \times 4 + 2$' stands for the number 14. Just as the symbol '14' is a name for the number 14, the expression '$3 \times 4 + 2$' is a name for the number 14. Naturally, a number can have many, many names. Some of the names are simpler in appearance than other names

and, frequently, the question of deciding which of two names for a number is the simpler is merely a matter of taste. Everyone would agree that, usually, the fewer marks needed to write an expression, the simpler the expression is. However, simplicity is seldom the most important thing about an expression. More important than simplicity is the use you intend to make of an expression. For example, compare the expressions for the number 231:

'3 × 7 × 11' and '231'.

Certainly '231' is a simpler expression than '3 × 7 × 11'. But if you wanted to know which numbers divided 231 exactly, the expression '3 × 7 × 11' would be more useful. Again, consider the two pairs of expressions:

(1) '$87\frac{11}{22}\%$' and '$\frac{7}{8}$'

and: (2) '$87\frac{6}{22}\%$' and '$\frac{48}{55}$'.

There is little doubt that most people would agree that '$\frac{7}{8}$' is simpler than '$87\frac{11}{22}\%$' and that '$\frac{48}{55}$' is simpler than '$87\frac{6}{22}\%$'. However, if we wanted to tell if $\frac{7}{8}$ were larger than $\frac{48}{55}$, we could reach a decision easier with the less simple expressions '$87\frac{11}{22}\%$' and '$87\frac{6}{22}\%$'.

EXERCISES

A. Write five different expressions for each of the listed numbers. For each exercise arrange the five expressions from the simplest to the most complicated.

1. 4 2. 13 3. -9 4. -8

B. The following is a list of expressions for numbers. Look at the first expression. It is a name for a certain number. Search through the list for all of the expressions which are also names of this number and write them in a

the

Mr. Dietz's class found this to be a very good review exercise. We suggest that you make more exercises of this type yourself. The students need this kind of practice in order to be able to "evaluate" expressions later in the unit.

* * *

Both Miss McCoy and Mr. Marston reported poor results for Part C on page 2-30 because of the inability of students to handle fractional numbers. Apparently, this is a good spot to give one or more extra assignments using the homework paper we have provided for this part, and a new set of replacements for the pronumerals.

column under the first expression. In this way, regroup the expressions into columns such that all the expressions in any column stand for the same number.

$5 \times 2 + 4$

$9 \times 1 - 6 \times 2$

$23 - 2 \times 3$

$\frac{7}{-.5}$

$\frac{2}{3}(21)$

$\frac{8}{1+1} + 5$

$(-3)(-4)$

$11 + 3$

$6(2 + 1)$

$\frac{40 + (-6)}{2}$

$6 \cdot 2 - 2 \cdot 13$

$2 + 3 \times 4$

7×2

$7(3 - 5)$

$.5(33 + 1)$

$-8 + 3(-2)$

$8 \times 2 + 1$

$50\% \times 28$

$\frac{30}{4} + \frac{19}{2}$

$[7(2)][-1]$

$2(5 \div 2)$

$-\frac{1}{3}(42)$

$\frac{29 + (-1)}{2}$

$(15 - 8)2$

$\frac{(9)(2) + (3)(6)}{5 - 8}$

$2 \times 5 + 2 \times 2$

$11 \cdot 2 - 16(\frac{1}{2})$

$(-7)(-2)$

$(6 + 1)(9 - 7)$

$87\frac{1}{2}\%\,(-16)$

$5(2 - 7) + 22$

$\frac{(13)(3) + (4)(3)}{3}$

(continued on next page)

$\frac{6 + 2(10 + 1)}{-2}$ $3 \times 1 + 3 \times 2$

$7(3 - 5)$ $5(-2) - 4$

$7 \cdot 2 + 3$ $(-8)(2) + 3(11)$

$\frac{-17}{2} - \frac{-11}{-2}$ $8(2 - \frac{7}{8})$

$2 + (6 - 1)(2 + 1)$ $-1 \cdot 5 + 2 \cdot 1$

$6(8 - 6) - 24$ $(2 + 5)3 - 3(6 + 5)$

$-\frac{3}{2}[1 + 1\frac{1}{2}(\frac{2}{3})]$ $-3(-4)$

$20\%(85)$ $[(2 + 5) \div 3] + 7$

$20 - 3$ $11 - 23$

$4\%(-300)$ $200 \times 7\%$

C. Each of the following expressions contains pronumerals. Replace the pronumerals by numerals and simplify.

'○' by '2' '□' by '-3'
'__' by '$\frac{1}{2}$' '⬡' by '$\frac{2}{3}$'
'△' by '$-2\frac{1}{3}$' '◇' by '-1.57'

1. ○ + ◇ 2. __ - □
3. __ + ◇ 4. □ + △
5. ○◇ 6. ○ □
7. □(△) 8. △△
9. 2○ + 4□ 10. 5△○ - 4◇
11. 2__△ - 3◇○ 12. 8△ + 5
13. -6△ △ + 2△ 14. △ ÷ □ + ⬡ ÷ __
15. $\frac{4 \bigcirc \square}{__ \diamond}$ 16. $\frac{3 \bigcirc \triangle + 2 \square \square}{5\triangle - 2__}$

e pages

We are very much interested in how students answer the question in Part D. If possible, send us their written responses. Notice that this is a repetition in new terms of the incorrect statement, 'In algebra you add, subtract, multiply, and divide letters'. Here is a place to discuss the use of colloquialisms [see T. C. 27A, B].

* * *

Section 2.05 contains the first practical application of pronumerals. The student should find that in order to make a generalization about numbers, he must use some form of pronumeral.

D. Suppose a younger brother or sister looked at these pages in your mathematics textbook and said,

> "Oh, I see. In algebra you really add and subtract and multiply and divide circles and spaces and triangles and diamonds and funny-shaped things like those instead of numbers."

How would you reply to this remark?

2.05 Pronumerals and principles of arithmetic.--In Unit I and in arithmetic you learned a great many facts about numbers. You learned addition facts, subtraction facts, multiplication facts, and division facts. Actually, it would have been impossible for you to learn so many facts individually. Instead, you learned certain principles which served to organize many facts into a few patterns. For example, consider the following multiplication facts.

$$1 \times 1 = 1$$

$$-2 \times 1 = -2$$

$$-3 \times 1 = -3$$

$$5 \times 1 = 5$$

$$7\tfrac{1}{2} \times 1 = 7\tfrac{1}{2}$$

$$-9.5 \times 1 = -9.5$$

Each of these facts appeared to be a specific application or an instance of a general principle. In Unit I, we called this the principle of one.

You will recall that in Unit I we did not state the principle of one but gave only illustrations of it. How would you state this principle?

Suppose you tried to state the principle by listing every instance of it. Your list might start with the examples we gave above. Could you ever finish the list, that is, could you ever say or write down every instance? It is easy

You may find the following explanation helpful.

The sentence

'x × 1 = x'

is neither true nor false. However, the double-boxed statement below is either true or false. (It happens to be true.)

Each of the following statements is true:

1 × 1 = 1,
2 × 1 = 2,
2.1 × 1 = 2.1,
7 × 1 = 7,
8.5 × 1 = 8.5,
987.03 × 1 = 987.03,
.
.
.

This double-boxed statement "containing" every possible application of the principle of 1 is precisely what is intended by the boxed statement on page 2-32.

* * *

The statement in the box on page 2-32 is <u>not</u> a statement of the principle of 1. Actually, it is a statement about the principle of 1, a statement which tells you that each instance of the principle of 1 is a true statement. [The boxed statement is a statement in the <u>metalanguage</u> which is the language used in talking about another language, the <u>object</u> <u>language</u>.] The principle of 1 is not stated until page 2-35.

to see that such a task is an impossible one. Moreover, it is not even necessary. For with pronumerals we can make a single statement which covers every instance of the principle of one.

You might jump to the conclusion that the statement we are seeking is the following:

$$\square \times 1 = \square$$

By itself, this expression is not enough. In fact if you look at it a moment it will appear silly. A fourth grader would probably translate it as, "a box times 1 equals a box." Remember, '$\square$' isn't the name of a number. It just holds a place in which you put a name of a number. And the only things you multiply are numbers.

We can use the expression:

$$\square \times 1 = \square$$

to speak about the principle of one if we say something else along with this expression.

> If we write in each '$\square$' of '$\square \times 1 = \square$' a numeral for any number, the resulting statement is true because it is an instance of the principle of one.

Since the phrase 'for any number' covers every case, the complete statement given above not only "summarizes" all of the specific facts you have already observed about multiplying by 1, but it tells you specific facts which you may not have considered. For example, it tells you that the following statements are true.

$$9,876,459 \times 1 = 9,876,459$$

$$-805\frac{43}{87} \times 1 = -805\frac{43}{87}$$

It is doubtful that you have ever considered these two instances.

The statement in the box stipulates, once and for all, the domain of the pronumerals used in most cases in FIRST COURSE. [Strictly speaking, our domain is the set of real numbers.] It is not necessary to single out the numbers of arithmetic because the numbers of arithmetic are isomorphic to the non-negative real numbers. As you proceed through the unit, you may find it necessary to remind students that when they make replacements for pronumerals, they should select negative numbers and 0 as well as positive numbers.

In our statement about the principle of one, we used the phrase 'for any number'. This phrase means, of course, that the principle applies to all of the numbers you have studied.

> Whenever we state a principle and we use either the phrase 'for any number' or the phrase 'for every number' we intend the statement to cover all directed numbers.

Sometimes we shall need to exclude certain numbers because the principle referred to does not apply to those numbers. In such cases, we shall state the exceptions. As an example, consider the following statement:

> If we write in each '$\square$' of '$\frac{\square}{\square} = 1$' a numeral for any number except 0, the resulting statement is true.

Why did we have to exclude 0? Do you get a true statement if you write '$5 - 2 \times 2\frac{1}{2}$' in each '$\square$'?

Another principle of arithmetic is illustrated by the following examples:

$$-3 + 0 = -3 \qquad 6 \times 0 = 0$$

$$5 + 0 = 5 \qquad -8 \times 0 = 0$$

$$-4\frac{1}{2} + 0 = -4\frac{1}{2} \qquad 7\frac{1}{2} \times 0 = 0$$

$$9.5 + 0 = 9.5 \qquad -3.8 \times 0 = 0$$

These facts are instances of the principle of zero. We tell how to use the principle as follows:

Miss Blair and Miss McCoy mentioned the need for stressing the fact that once a number is selected to provide replacements for pronumerals in an expression, each time that pronumeral occurs in the expression it must be replaced by a numeral for the selected number.

Students should check these statements by carrying out the instructions in the "if parts", and seeing if the "then parts" are satisfied.

* * *

When the student sees the expression '$\bigcirc + \square = \square + \bigcirc$', his first reaction ought to be one of anticipation. He should expect to be told something about the kind of symbols which are to replace the pronumerals. It is only after he has been told something about the replacement symbols that he can say that the resulting replacement instance is either true or false. He should never come to see the pronumerals as symbols for numbers; pronumerals are frames in which number symbols can be written.

Professor Meserve suggests that you stress the point that the selection of the pronumeral used in the statements in the boxes is quite arbitrary. For example, you should point out that one could write for a first statement in the top box:

If we write in each '$\bigcirc$' of

'$\bigcirc + 0 = \bigcirc$'

a numeral for any number, the resulting statement is true.

[But, of course, once '$\bigcirc$' is selected it must be used in all three places in the statement.] To emphasize this point, you might ask students to restate the boxed statements using different pronumerals.

* * *

(continued on T. C. 34B)

1. If we write in each '□' of

'□ + 0 = □'

a numeral for any number, the resulting statement is true because it is an instance of the principle of zero.

2. If we write in '◇' of

'◇ × 0 = 0'

a numeral for any number, the resulting statement is true because it is an instance of the principle of zero.

[Was it necessary to change the pronumeral symbol from '□' to '◇' in going from the first sentence to the second sentence?]

THE COMMUTATIVE PRINCIPLE FOR ADDITION

If in the expression

'○ + □ = □ + ○'

we write in each '○'

a numeral for any number

and we write in each '□'

a numeral for any number, the resulting statement is true because it is an instance of the commutative principle for addition.

classroom technique is that the students can recognize quickly that A and B can act completely independently of each other in choosing replacements. In fact A and B may pick the same replacement; the "double quantifier" does not compel them to select two numbers.

This practice in producing instances of generalizations will pay off when the notion of a counter-example to a generalization is introduced on page 2-37.

The generalization:

(9) For every ◯ ,

for every □ , □ + ◯ = ◯ + □ ,

which could be abbreviated:

(10) For every ◯ and □ , □ + ◯ = ◯ + □ ,

differs from (6) in that a typical instance of (9) is:

for every □ , □ + 3 = 3 + □ .

However, the instances of the instances of (9) are the same as the instances of the instances of (6). In particular, each instance of (10) is a consequence of (6), so (10) is a consequence of (6). Similarly, any instance [like (10)] of (9) is a consequence of (6), so (9) itself is a consequence of (6). Since, also, (6) is a consequence of (9), (6) and (9) are <u>equivalent</u>.

are equivalent equations, reconsider the generalization:

(1) For every $\square$, $\square$ $+ 0 = 0$,

and ask the class what would happen if student A called out '$\frac{1}{0}$' [or: '$\frac{1}{5-5}$']? Student B should say, "What number is that?" Student A has not really selected a number from the domain of '$\square$'. He has just mumbled some nonsense which sounded very much like a numeral but which wasn't a numeral at all.

Next, consider universal generalizations such as:

(5) For every $\square$ and $\bigcirc$, $\square + \bigcirc = \bigcirc + \square$.

Statement (5) is an abbreviation for:

(6) For every $\square$,

for every $\bigcirc$, $\square + \bigcirc = \bigcirc + \square$,

so its instances are universal generalizations such as:

(7) for every $\bigcirc$, $4 + \bigcirc = \bigcirc + 4$.

An instance of (7) is:

(8) $4 + 3 = 3 + 4$.

Statement (7) is a consequence of (6), and (8) is a consequence of (7). Hence, (8) is also a consequence of (6). Although, strictly speaking, (8) is not an instance of (6), it is sometimes convenient to call it so in order to avoid frequent repetition of such awkward statements as '(8) is an instance of an instance of (6)'. However, students must be aware that if they say, '(8) is an instance of (6)', they should have in mind that (8) is actually an instance of an instance of (6).

Students should practice obtaining instances of instances of (6) [or of (5)]. Student A calls out a replacement for '$\square$', student B then calls out a replacement for '$\bigcirc$', after which student C writes the corresponding instance. One of the advantages of this

(continued on T. C. 35H)

and, in particular,

> (iii) every sentence which has a false consequence is false.

Returning to (1) and (2), we label (1) 'true' because it follows from the definition for addition of directed numbers. Hence, by (i) and (ii), '3 + 0 = 3' is true. Consequently, '3 + 0 ≠ 3' is false. So, by (i) and (iii), (2) is false.

The rule (i) is part of a definition of the notion of logical consequence. We shall have more to say about this notion later, particularly in SECOND COURSE.

Students should also consider universal generalizations such as:

(3) For every $\square$ except 0, $\frac{\square}{\square} = 1$.

In this case, if student A calls out '0', B refuses to write, because 0 does not belong to the domain of $\square$, as this domain is specified by the "restricted quantifier" 'For every $\square$ except 0'. On the other hand, for the generalization:

(4) For every $\square$, $\frac{\square}{\square} = 1$,

when A calls out a numeral for 0, B writes the instance:

$$\frac{0}{0} = 1$$

of (4). This instance is a consequence of (4), but, since '$\frac{0}{0}$' is not a name for anything, it is false. So (4) is also false.

As preparation for explaining in Unit 3 the circumstances under which the equations:

$$x = 7, \qquad \text{and:} \qquad x + \frac{1}{x - 7} = 7 + \frac{1}{x - 7}$$

(continued on T. C. 35G)

[or: each instance of a universal generalization is a consequence of it (or: follows from it)].

There is nothing subtle about the assertion made in (i). It is the case merely because this is what we intend to convey when we write 'for every ---'. However, it is of major importance that one realizes that (i) refers to every universal generalization, "false" ones as well as "true" ones. For example, the universal generalization:

(2) For every $\square$, $3 + \square \neq 3$

implies each of its instances, despite the fact that one of these instances, '$3 + 0 \neq 3$', contradicts an instance, '$3 + 0 = 3$', of (1). Since our beliefs concerning directed numbers predispose us to an unquestioned acceptance of (1), we must [by (1)] accept its instance '$3 + 0 = 3$', and so reject '$3 + 0 \neq 3$'. Since the latter is a consequence of (2), we must [again by (i)] also reject (2).

Statements which we accept [on whatever ground] we customarily call 'true'. You use many methods in deciding whether or not statements are true. You may see Lake Michigan a mile away, and decide that 'I am a mile from Lake Michigan' is true [you may be wrong; perhaps you are seeing a mirage]; you may feel that you can't understand mathematics, and decide that 'I can't understand mathematics' is true [you may be wrong; perhaps you haven't given yourself a chance]. Another way which you use to decide whether a sentence is true is to find out whether it is a consequence of sentences which you have already decided are true. For example, the sentence:

I can walk to Lake Michigan in 20 minutes

is a consequence of the sentences:

I am a mile from Lake Michigan,

and:

I can walk a mile in 20 minutes.

Hence, if you classify the second and third of these sentences [premisses] as true, you will also decide that the first sentence [conclusion] is true. [In deciding, in this way, whether a sentence is true, you may be wrong in either of two ways: one of the premisses may not be true, or the conclusion may not be a consequence of the other two.] In general,

(ii) every consequence of true sentences is true,

(continued on T. C. 35F)

For these reasons the use of quantifiers such as 'for every $\square$ ' and 'there is (or: exists) a number x such that' is to be preferred.

[Phrases which contain 'any' appear, at first glance, to be quantifiers. In:

If the price is right, anyone will buy it,

'anyone' can be replaced by the universal quantifiers 'everyone'. But, in:

I shall be surprised if anyone buys it,

'anyone' takes the place of an existential quantifier. This ambiguity of usage suggests that 'anyone' is not a quantifier and, indeed, it is not. Phrases which contain 'any' are variables, and sentences in which they occur are open sentences, analogues of such sentences as ' $\square \times 1 = \square$ '. Thus, some open sentences of the English language are statements. This is in contrast to our present interpretation of open sentences in FIRST COURSE, in which an open sentence is neither true nor false. In later courses we shall introduce the practice of treating open sentences, on occasion, as statements. For the present, this (including the use of 'any') should be avoided.]

* * *

It is instructive to spend a little time in class learning how to generate instances of a universally quantified sentence. Consider the principle of 0 for addition:

(1) For every $\square$, $\square + 0 = 0$.

We know, by convention, that replacements for ' $\square$ ' in ' $\square + 0 = 0$' are names of directed numbers. So, <u>each</u> directed number gives an instance of generalization (1). Select two students, A and B. A's job is to pick numbers from the domain of ' $\square$ ' [use the word 'domain'] and "call" them out. B's job is to listen to A and to write that instance of (1) which corresponds to each number A selects. The class notes that, for as long as they accept (1), they are committed to accepting each of its instances. This is because

(i) each universal generalization <u>implies</u> each of its instances

(continued on T. C. 35E)

first of these two meanings is stated unambiguously in:

(6_1) There is a ☐ such that, for every ⬭, ☐ loves ⬭,

and the second is:

(6_2) For every ⬭, there is a ☐ such that ☐ loves ⬭.

Sentence (6_1) is an existential generalization whose instances are universal generalizations, such as:

(7_1) For every ⬭, John loves ⬭,

while (6_2) is a universal generalization whose instances are existential generalizations, such as

(7_2) There is a ☐ such that ☐ loves Joan.

Phrases like: somebody, everybody, for every number x, for each ☐, and: there exists a ◯ such that, are called quantifiers. The first and last are existential quantifiers, the others are universal quantifiers. Each quantifier refers in some way to the members of some set, the domain of the quantifier. For example, in 'somebody', 'body' refers to persons, and 'for each ☐', by our convention that '☐' is a pronumeral, refers to numbers. The occurrence of a quantifier in a sentence indicates that some property is being considered to be generalized, universally or existentially, over the domain of the quantifier. For example, 'Somebody loves Joan' states that the property of loving Joan is existentially generalized over the set of all persons, while 'John loves everybody' states that the property of being loved by John is universally generalized over the same set. The trouble with (6′) is that one may not be sure whether it states that the property of loving everyone is existentially generalized [as does sentence (6_1)], or that the property of being loved by someone is universally generalized [as does sentence (6_2)]. Another way of putting the difficulty is that one may not be sure whether 'John loves everybody' [(7_1)], or 'Somebody loves Joan' [(7_2)], is an instance of (6′).

(continued on T. C. 35D)

The letter 'X' may facetiously be called 'a pronumeralal'; just as a pronumeral, such as ' ☐ ', holds a place for names of numbers, 'X' holds places for names of numerals (which might be called 'numeralals'). A more common terminology than 'pronumeralal' is: variable whose domain is the set of all numerals. (Variables, like these, which are used in discussing the syntax of a language are often called 'syntactic variables'.)

As a non-mathematical analogy, consider:

(3) If in the expression ' ☐ is mortal' we write in ' ☐ ' a name for any person, the resulting statement is true.

This implies, for example, that:

(4) John is mortal

is true. We must accept (3) [and, hence, we must also accept (4)] if we accept the generalization:

(5) For every person ☐ , ☐ is mortal.

In colloquial English, the fact about people which is expressed by (5) might be stated as:

(5′) Everybody is mortal.

There are two difficulties with this latter mode of speech. In the first place, although (5′) is a generalization, it looks like the subject-predicate sentence (4), and this confuses the essential distinction between generalizations and other kinds of sentences. The resulting confusion, for example, may engender a belief in a "general person" (c.f. 'general number') about whom the sentence speaks. In the second place, this mode of speech is inadequate for expressing slightly more complicated ideas. For example:

(6′) Somebody loves everybody

is ambiguous. It may mean that there is some person who loves everyone, indiscriminately, or it may mean that no one is unloved. The

(continued on T. C. 35C)

The box on page 2-32 contains a rule for constructing statements such as:

(1) $3 \times 1 = 3$, $(-7.5) \times 1 = -7.5$, etc.

Each of these statements is an instance of another statement about numbers, the principle of one:

(2) For every $\square$, $\square \times 1 = \square$.

[Statements, like (2), which assert that everything of some kind has a certain property are called universal generalizations. Statements like 'There is a $\square$ such that $2 + \square = 2$' are called existential generalizations.] Since a universal generalization implies each of its instances, each of the sentences (1) is a logical consequence of sentence (2). Since (2) is a true statement about numbers, it follows that each of statements (1) is a true statement about numbers. It is important to notice that (2) is a statement in which pronumerals are used in order to express a fact about numbers, while in the boxed statement on page 2-32 we speak about pronumerals in order to assert a fact about numerals (the fact that if we combine numerals in a certain way we get true statements about numbers).

[The boxed statement (2-32) can be rephrased in such a way as to make clear that it is a universal generalization about numerals:

For each numeral X, if, in the expression '$\square \times 1 = \square$', we write X in each '$\square$', then the resulting statement is true because it is an instance of the principle of one.

An instance of this generalization is:

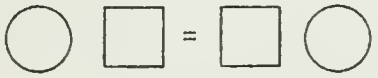
if, in the expression '$\square \times 1 = \square$', we write '3' in each '$\square$'; then the resulting statement is true because _____.

This asserts that '$3 \times 1 = 3$' is true because _____.

(continued on T. C. 35B)

THE COMMUTATIVE PRINCIPLE FOR MULTIPLICATION

If in the expression

'  '

we write in each ' ' a numeral for any number and we write in each ' ' a numeral for any number, the resulting statement is true because it is an instance of the commutative principle for multiplication.

Notice that our statements about the principles are rather long and cumbersome. This is because we have been telling how to obtain instances of the principles. Now, we shall use pronumerals to state the principles themselves. For a time we shall state how to obtain instances and also state the principles. Here are statements of principles which have been mentioned above.

The Principle of One

For every □, □ × 1 = □.

The Principle of Zero

1. For every □, □ + 0 = □.
2. For every □, □ × 0 = 0.

The Commutative Principle for Addition

For every □ and ○, □ + ○ = ○ + □.

The Commutative Principle for Multiplication

For every □ and ○, □○ = ○□.

Because you accept these principles as true you will get true instances when you substitute numerals for the pronumerals in the parts of the statements following the commas. Notice that it doesn't make sense if you substitute numerals for

by the conjunction of the other and '$1 \times 0 = 0 \times 1$' [and that the last is a consequence of the commutative principle for multiplication].

* * *

To make sure that the students understand how to use the universally quantified statements from Exercise 13 on, ask them to describe <u>how</u> each generalization can be used to produce instances.

Sample 2 requires more explanation. The statement:

(1) For every $\bigcirc$, $\bigcirc \times 3 = 3 \times \bigcirc$

is a consequence of the commutative principle for multiplication because each instance of (1) is also an instance [and, hence, a consequence] of this principle. In general, a universal generalization is implied by each sentence which implies every instance of the generalization. [For example, the universal generalization (1) is implied by the commutative principle for multiplication.] Since we have decided that multiplication of directed numbers is commutative, we must accept consequences of the commutative principle, such as (1), and label such consequences 'true'.

It is incorrect, however, to refer to (1) as an <u>instance</u> of the commutative principle for multiplication, as is suggested by the instructions for Part A. Also, although the solution of Sample 2, given in the paragraph above, is an <u>example</u> of the use of the commutative principle, it seems scarcely proper to refer to (1) itself as an <u>example</u> of this principle, as is done in the Solution in the text. One should, rather, speak of (1) as a <u>case</u> of the commutative principle for multiplication. In general, one universal generalization is a case of another if each instance of the first is also an instance of the second. As noted in the preceding paragraph, a universal generalization implies each of its cases. Consequently, each case of a <u>true</u> universal generalization is true.

* * *

The statement in Exercise 11 is an instance of the principle of 1, while that of Exercise 12 is an instance of the principle of 0 for multiplication. Note also that each of these statements is implied

(continued on T. C. 36B)

pronumerals throughout the entire statement or principle, or even in 'for every □' or 'for every □ and ○'.

EXERCISES

A. The following statements are true. Of which principle is the statement an instance?

Sample 1. $-3 + 5 = 5 + (-3)$

Solution. This is an instance of the commutative principle for addition in which one pronumeral has been replaced by '-3' and the other pronumeral has been replaced by by '5'.

Sample 2. For every ○, ○ × 3 = 3○.

Solution. This is an example of the commutative principle for multiplication in which one of the pronumerals has been replaced by '3'.

1. $3 \times 1 = 3$
2. $4 + 0 = 4$
3. $3.23 \times 0 = 0$
4. $\frac{2}{3} + \frac{1}{5} = \frac{1}{5} + \frac{2}{3}$
5. $427 \times 92 = 92 \times 427$
6. $3.004 + 0 = 3.004$
7. $4{,}000.002 \times 1 = 4{,}000.002$
8. $1 \times 1 = 1$
9. $0 + 0 = 0$
10. $0 \times 0 = 0$
11. $0 \times 1 = 0$
12. $1 \times 0 = 0$
13. For every ⬡, ⬡ × (2 - 1) = ⬡.
14. For every △, △ + (42 - 42) = △.
15. For every ◇, ◇ + 5 = 5 + ◇.

A calls out a replacement for ' ◯ ', B calls out a replacement for ' □ ' so that, together the two replacements produce a true sentence.

Ask your students how it is possible for B to do this. [Suppose A calls out '5'; then B calls out '5', also. Etc.] Then ask how A can defeat B by making B call out a replacement which will show that A's replacement produces a false instance. [In this "replacements-instance" game, A must call out first, and B must call out second. (A gives first component, and B gives second component.) Under ordinary circumstances, it would often happen that A would select the same first component several times while B would select different second components. For example, you might get from A and B the ordered pairs (9, 3), (9, 8), and (9, 82.5). So, A decides to capitalize upon his right to repeat his selection in order to defeat B. He calls out '5', and then B calls out '5' thereby producing a true instance. Then A calls out '5' again. B follows with '5'. A protests, saying, "You know the rules. You can't repeat a pair." B, who hates to give up, retracts the '5', and calls out instead, '4 + 1'. A replies, "I know my arithmetic. 4 + 1 = 5. You're still repeating what we had before." B gives up, and calls '3791.8' which shows that A's '5' gives a false instance.]

* * *

The generalization in Exercise 1 is a case of the commutative principle for addition [each instance of the generalization in Exercise 1 is an instance of the commutative principle for addition], and, so, is a consequence of that principle. Since the latter generalization is true, the generalization in Exercise 1 is also true.

The very important idea of disproving a generalization by exhibiting a counter-example is introduced in the Sample of Part B. To say that the generalization stated in the Sample is true is to say that each of its instances is true. Thus, if a false instance can be exhibited, the generalization is not true, that is, the generalization is false. [Note that just one false instance is enough.] It is important to point out here that the counter-example is the number which produces the false instance, and that it is not the false instance itself. The instance of the Sample corresponding to the choice of '5' for '◯' is:

(1) for every □, 5 - □ = □ - 5,

and this generalization is false because it has the false instance

(2) 5 - 3 = 3 - 5.

Since the instance (1) of the Sample is false, the Sample itself is false. A number which yields a false instance of a universal generalization is called a counter-example to the generalization. Thus, the number 5 is a counter-example to the Sample. [Note that it is the number 5, rather than the corresponding instance (1), which is the counter-example.

Just as students may be allowed to say that (2) is an instance of the Sample [although it is actually an instance of an instance of the Sample], they may say that the ordered pair whose first component is 5 and whose second component is 3 is a counter-example to the Sample. By this they should mean that 5 is a counter-example to the Sample because 3 is a counter-example to the instance of the Sample which corresponds to 5.

An instructive classroom gimmick is the following problem. Consider the false generalization in the Sample. Both A and B know that the generalization is false, but B decides to be uncooperative. Whenever

(continued on T. C. 37B)

16. For every ◯, 493 + ◯ = ◯ + 493.

17. For every □, □ + 1 = 1 + □ .

18. For every ____, _____ × 1 = 1 × _____.

19. 1 × 5 = 5 [Note: This is not a direct instance of the principle of one. You need two principles to assure you that it is true.]

20. $1 \times \frac{372}{499} = \frac{372}{499}$

21. For every △, 6△ = △ × 6.

22. For every ⬡, ⬡ + 0 = 0 + ⬡.

23. For every ◯, $\frac{1}{8}$◯ = ◯ × $\frac{1}{8}$.

B. Some of the following statements are true and some are false. If a statement is true, give the principle which justifies the statement. If the statement is false, give an example where replacing the pronumerals by numerals gives a false statement.

Sample. For every ◯ and □, ◯ - □ = □ - ◯.

Solution. This statement is false because replacing, for example, '◯' by '5' and '□' by '3' gives:

5 - 3 = 3 - 5

which is a false statement.

1. For every ◯ and □, 2◯ + □ = □ + 2◯.

2. For every ◯ and □, 2◯ + □ = 2□ + ◯.

3. For every △ and ⬡, △ ÷ ⬡ = ⬡ ÷ △.

4. For every △, △ + 0 = 0.

5. For every , + 1 = × 1.

6. For every _____, _____ × 1 = 1.

7. For every , + 1 = .

THE ASSOCIATIVE PRINCIPLE FOR ADDITION

We tell how to obtain instances of this principle and then state the principle.

If in the expression

'(○ + △) + □ = ○ + (△ + □)'

we write in each ' ○ ' a numeral for any number, and

we write in each ' △ ' a numeral for any number, and

we write in each ' □ ' a numeral for any number,

the resulting statement is true because it is an instance of the associative principle for addition.

For every ○ , △ , and □ ,

(○ + △) + □ = ○ + (△ + □).

THE ASSOCIATIVE PRINCIPLE FOR MULTIPLICATION

If in the expression

'(○ △) □ = ○ (△ □)'

we write in each ' ○ ' a numeral for any number, and

we write in each ' △ ' a numeral for any number, and

we write in each ' □ ' a numeral for any number,

the resulting statement is true because it is an instance of the associative principle for multiplication.

For each △, □, and ◇,

△□ + △◇ = △(□ + ◇).

Because of the incorrect notions about equality which he may have picked up in elementary school, a student may need a certain amount of re-education before he recognizes the symmetry of equality.

Whenever a student errs in applying one of the principles, he should be asked to state the principle (his statement should be written on the board) and to tell what replacements he used to produce the instance in question. This procedure will reinforce the idea that the process of transforming expressions is based not on magic but on a relatively small number of principles.

Here are illustrations of possible justifications for several of the exercises.

*

Exercise 5

$$\begin{aligned} 7(2+5) &= 7\times 2 + 7\times 5 && \text{[distributivity]} \\ &= 14 + 35 && \text{[facts of arithmetic]} \end{aligned}$$

*

Exercise 9

$$\begin{aligned} 24 + 6 &= 3\times 8 + 3\times 2 && \text{[facts of arithmetic]} \\ &= 3(8+2) && \text{[distributivity]} \end{aligned}$$

This approach is somewhat "unnatural" but should be demonstrated because, frequently, in constructing derivations we know the end-product and use that information to guide us. Also, some students need to be shown that:

$$3\times 8 + 3\times 2 = 3(8+2)$$

is a consequence of the distributive principle. Such students seem to be able to use the distributive principle in expanding, but are unable to see its application in factoring. Indeed, it may be instructive to restate the distributive principle this way:

(continued on T. C. 39C)

The exercises in Part A are designed to give practice in applying the principles of arithmetic, and in becoming aware of certain other principles which can be derived from them. Before a student attempts to justify any of the exercises he should first determine whether he has a true statement. Now, a student could (and should) argue that this preliminary screening is all that is needed in justifying the true statements. In other words, if we know enough facts of arithmetic [such as the results expressed in addition and multiplication tables, or obtained by the use of the algorithms for addition and multiplication] to convince ourselves that some of the statements are false, then we also know enough to convince ourselves (and anyone else) that the others are true. [After all, the statements are <u>not</u> generalizations!] But this is not the point of the exercises. The point is to take one of the sides of a true equation and transform it into the other side of that equation. [Use the word 'transform' in class.] And the trick is to do this using as few of the facts of arithmetic as possible.

Every once in a while, a student will use a principle which he is willing to accept but which has not been explicitly stated. There are two things which can be done at this point, Ask the class to state this principle (this means with proper quantification) and either agree to accept it, or try to derive the stated principle from the other principles. [Since we are not developing a deductive theory with carefully stated postulates and faultlessly deduced theorems, a certain amount of informality (i.e., arm-waving) is in order.] With a few exceptions, the derived principles are not important enough to name and keep a record of, but if some students want to keep such a record (statements and derivations) they should be encouraged to do so. At least, they can write the principles on the facing pages.

(continued on T. C. 39B)

For every ○ , △ , and □ ,

(○ △) □ = ○ (△ □) .

THE DISTRIBUTIVE PRINCIPLE

If in the expression

' △ (□ + ◇) = △ □ + △ ◇ '

we write in each ' △ ' a numeral for any number, and

we write in each ' □ ' a numeral for any number, and

we write in each ' ◇ ' a numeral for any number,

then the resulting statement is true because it is an instance of the distributive principle.

For every △ , □ , and ◇ ,

△ (□ + ◇) = △ □ + △ ◇ .

EXERCISES

A. The following statements are either true or false. For each true statement give the principle (or principles) which justify it.

1. $(5 \times 3) \times 2 = 5 \times (3 \times 2)$

2. $(3 + 4) + 2 = 3 + (4 + 2)$

3. $6 \times 1 = 6$

4. $1 \times 5 = 5 \times 2$

5. $7(2 + 5) = 14 + 35$

6. $8 \times (9 \times 2) = 72 \times 2$

7. $6(3 + 4) = 63 + 64$

8. $(8 \div 2) + 5 = 8 + 7$

9. $24 + 6 = 3(8 + 2)$

10. $100 + 50 = 50(2 + 1)$

11. $1 \times 4 = 4 \times 1$

12. $4 \times 1 = 1$

or:

$$\triangle = \triangle\,(\,\triangle \div \square + \diamond\,)$$

or:

$$\triangle = \bigcirc \quad \text{or} \quad \triangle + \square + \diamond = 1$$

are the only ordered triples which produce true instances.]

A demonstration for Exercise 30 is easier to construct if one starts with the expression at the right of '='.

$$\left(\tfrac{1}{2} + \tfrac{1}{3}\right)\left(\tfrac{1}{2} + \tfrac{1}{6}\right)$$

$$= \left[\tfrac{1}{2}\left(\tfrac{1}{2}\right) + \tfrac{1}{3}\left(\tfrac{1}{2}\right)\right] + \left[\tfrac{1}{2}\left(\tfrac{1}{6}\right) + \tfrac{1}{3}\left(\tfrac{1}{6}\right)\right] \qquad \text{[Exs. 21 and 22]}$$

$$= \left[\tfrac{1}{2}\left(\tfrac{1}{2}\right) + \tfrac{1}{3}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{6}\right)\right] + \tfrac{1}{3}\left(\tfrac{1}{6}\right) \qquad \text{[associativity for addition]}$$

$$= \left[\tfrac{1}{2}\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{3}\right) + \tfrac{1}{2}\left(\tfrac{1}{6}\right)\right] + \tfrac{1}{3}\left(\tfrac{1}{6}\right) \qquad \text{[commutativity for multiplication]}$$

$$= \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6}\right) + \tfrac{1}{3}\left(\tfrac{1}{6}\right) \qquad \text{[Ex. 25]}$$

$$= \tfrac{1}{2}\left(1\right) + \tfrac{1}{3}\left(\tfrac{1}{6}\right) \qquad \text{[fact of arithmetic]}$$

$$= \tfrac{1}{2} + \tfrac{1}{3}\left(\tfrac{1}{6}\right) \qquad \text{[principle of 1]}$$

* * *

The exercises in Part B should be done orally. We are trying to build in the student the ability to recognize sentences which when quantified will produce true generalizations. He should develop speed in recognizing the key principle or principles which support such generalizations, and speed in exhibiting counter-examples for the false generalizations.

of distributivity in connection with a pair of operations. One operation may or may not have the property of being distributed over the other. For example, the operation of multiplication can be distributed over the operation of addition. That is, the product of a given number by a given sum is equal to the sum of the products of the given number by each of the addends of the given sum. Similarly, the operation of multiplication can be distributed over the operation of subtraction. Now, Exercises 29 and 30 suggest the question: Can the operation of addition be distributed over the operation of multiplication? [The suggestion of this question would have been made more forceful if the exercises had been written:

$$5 + (3 \times 4) = (5 + 3) \times (5 + 4)$$

and:

$$\frac{1}{2} + (\frac{1}{3} \times \frac{1}{6}) = (\frac{1}{2} + \frac{1}{3}) \times (\frac{1}{2} + \frac{1}{6}).$$]

In other words, is it the case that

for every △ , □ , and ◇ ,

△ + (□ ◇) = (△ + □)(△ + ◇) ?

Since Exercise 29 is a false instance of this generalization, the answer is 'No'. But, surprisingly enough, the generalization does have true instances, and Exercise 30 is one of the true instances. [Your able students should be challenged by a request to find more true instances of this false generalization. The ordered triples (△ , □ , ◇) which satisfy:

△ + (□ ◇) = (△ + □)(△ + ◇)

(continued on T. C. 40L)

of the next day's class, "I have another way to do Exercise 27."

$4(9 - 3)$	$= 4[9 + (-3)]$	[subtraction principle]
	$= 4(9) + 4(-3)$	[distributivity]
	$= 4(9) + 4[(-1)3]$	[principle of -1]
	$= 4(9) + [4(-1)]3$	[associativity of multiplication]
	$= 4(9) + [(-1)4]3$	[commutativity of multiplication]
	$= 4(9) + (-1)[4(3)]$	[associativity of multiplication]
	$= 4(9) + \{-[4(3)]\}$	[principle of -1]
	$= 4(9) - 4(3)$	[subtraction principle]

"I used the principle of -1, 'For every $\square$, $-1 \times \square = -\square$ ' [see Part D on page 1-55, Unit 1]."

* * *

Students should recognize that Exercise 28 involves an instance of the same principle of which Exercise 27 was an instance. Thus, the same demonstration would have to be repeated for Exercise 28. It is time to state the principle in question, and add it to our kit of principles of arithmetic.

For every $\triangle$, $\square$, and $\diamond$,

$$\triangle(\square - \diamond) = \triangle\square - \triangle\diamond .$$

Call it 'the distributive principle for subtraction' and when citing it to support an instance simply write 'distributivity'.

* * *

Exercises 29 and 30 provide an opportunity for taking a second look at the distributive principle and its subtraction version. One thinks

(continued on T. C. 40K)

We use part 1 to support '3 + (-3) = 0', and part 2 to support 'if 4(3) + 4(-3) = 0 then 4(-3) = -[4(3)]'. Our transformation proceeds as follows.

4(9 - 3)	= 4[9 + (-3)]	[principle of subtraction]
	= 4(9) + 4(-3)	[distributivity]
	= 4(9) + {-[4(3)]}	[principle of opposites, (2)]
	= 4(9) - 4(3)	[principle of subtraction]

So, we have a demonstration with justifications based on general principles rather than on specific facts of arithmetic. The student has not given a derivation of his proposed "distributive principle for subtraction", but he is sure that such a derivation would follow very closely the derivation given above because the recourse to special facts has been eliminated. The student has experienced some of the pleasures which result from the mathematician's search for generality. The student has not done much in trying to organize principles into a "deductive sequence". But as he continues to state principle after principle he will begin to evince the same dissatisfaction with this unwieldly accumulation of principles as he did with his large mass of facts of arithmetic. That will be the time for the student to begin work on constructing a deductive theory in which just a few of the principles are designated 'postulates' and the rest are deduced as theorems from the postulates. Before the student is ready for formal work with a deductive theory, he must feel very much at home with generalizations, and must be able to give precise verbalizations of generalizations. Clearly, there is need for much exploratory work. And that is one of the major purposes of FIRST COURSE!

To cap this account of how some FIRST COURSE classes handled Exercise 27, we tell about one student who remarked at the opening

(continued on T. C. 40J)

One student tries:

$$\begin{aligned} 4(3) + 4(-3) &= 4[3 + (-3)] && \text{[distributivity]} \\ &= 4(0) && \text{[principle of opposites]} \\ &= 0 && \text{[principle of 0 for multiplication]} \end{aligned}$$

So, the students claim, the principle of opposites tells us that 4(-3) and 4(3) are opposites because their sum is 0.

"But what is the principle of opposites? Who can state it?" This seems to be easy for the class, especially in view of the "instance" above: 3 + (-3) = 0. Bill says:

For every $\square$, $\square + (-\square) = 0$.

Teacher objects, "I can see how this tells us that 3 + (-3) = 0, but I can't see how it tells us that 4(3) and 4(-3) are opposites. Can anyone show that '4(3) + 4(-3) = 0' is an instance of the principle Bill stated, a principle which he claimed is the principle of opposites?" No one can, because '4(3) + 4(-3) = 0' is not an instance of Bill's generalization.

It is clear now that the class needs to reconsider the principle of opposites. So, we ask how the principle is used. For one thing, we use the principle to tell us that if two numbers are opposites, their sum is 0. [Instance: if 3 and -3 are opposites then 3 + (-3) = 0. We know 3 and -3 are opposites because of the previously stated principle: For every $\triangle$, $-\triangle$ is the opposite of $\triangle$.] But we also use the principle to tell us that if the sum of two numbers is 0, the numbers are opposites. So, <u>the principle of opposites</u> seems to have two parts:

1. For each $\square$, $\square + (-\square) = 0$,
2. For each $\square$ and $\triangle$, if $\square + \triangle = 0$ then $\triangle = -\square$.

(continued on T. C. 40I)

"Oh, we don't mean it that way. We mean if you put a number, er, I mean a numeral, in the triangle, and then put a minus sign in front of the triangle, you get the opposite of the number."

"O.K. Who can state what she just said as a principle?" Several students can; we get:

For every △ , - △ is the opposite of △ .

So, now we have a principle, 'For every □ and △ ,

□ - △ = □ + (- △)', which we can add to our list of principles. We decide to call it 'the principle of subtraction'. [We do not derive it at this time, nor is there any demand from the students for a derivation. There will be opportunities later [see page 2-60] to stimulate such a demand.] We continue with the transformation.

$$\begin{aligned} 4(9 - 3) &= 4[9 + (-3)] && \text{[principle of subtraction]} \\ &= 4(9) + 4(-3) && \text{[distributivity]} \end{aligned}$$

What next? There now arises the suggestion that 4(-3) is the opposite of 4(3), and so we can use the principle of subtraction to produce the desired expression. How do you know that 4(-3) is the opposite of 4(3)? No response. The teachers tries to be "helpful": It's easy to see that 4(-3) is the opposite of 4(3). Just work them out. 4(-3) is -12, and 4(3) is 12, and -12 is the opposite of 12. A few students nod in agreement. But most are not satisfied. There ought to be an <u>easier</u> way to do it, they say. [Translate 'easier' as 'more elegant' or 'more general'.]

(continued on T. C. 40H)

had explored some connections between addition and subtraction, and they sensed that the proposed principle ought to be derivable from the distributive principle and the connections between addition and subtraction. In other words, they sensed that there were enough principles available to them at the verbal level, and enough principles of which they were aware only at the non-verbal level [but which could now be verbalized] to enable them to derive the proposed principle, or at least to reduce the number of "fact of arithmetic" citations in carrying out the transformation.

The first step in the transformation:

$$4(9 - 3) = 4[9 + (-3)]$$

clearly involves an instance ['9 - 3 = 9 + (-3)'] of a generalization (principle) which students used repeatedly in Unit 1. Some of the students had verbalized this principle in Unit 1 as their short-cut rule: to subtract a number just add its opposite. Now, if this is a principle, we ought to be able to use pronumerals to give a precise formulation. And, so, a student volunteers:

(1) For every $\square$ and $\triangle$,

$$\square - \triangle = \square + (- \triangle).$$

We write this on the board. To ensure that the class understands the symbolism, we ask, "What is this?", pointing to '$(- \triangle)$'.

"That's the opposite of triangle," several students reply in chorus.

"Wait a minute," the teacher counters, "how can a triangle have an opposite?"

(continued on T. C. 40G)

fact that 9 - 3 = 9 + (-3) are also facts of arithmetic, that is, facts of the arithmetic of directed numbers. [Moreover, because of the isomorphism between the numbers of grade school and non-negative directed numbers, the facts of grade school arithmetic are "mirrored in" the facts of arithmetic (of directed numbers).]

* * *

Exercise 27 provided teachers at University High School with an opportunity to do a bit of creative work with their classes. After following the demonstration given above, students objected, claiming that it lacked elegance and generality. Of course, they didn't use these words, but they did see that the procedure was of the same nature as the following.

4(9 - 3)	= 4(6)	[fact of arithmetic]
	= 24	[fact of arithemtic]
	= 36 - 12	[fact of arithmetic]
	= 4(9) - 4(3)	[facts of arithmetic]

The students were sure that the statement in Exercise 27 was an instance of a true generalization; they even supplied a name for this generalization:

the distributive principle for subtraction.

Some students supplied this statement of the principle:

For every △ , □ , and ◇ ,

△ (□ - ◇) = △ □ - △ ◇ .

Now, we could have accepted this principle, after verifying it in a few cases, just as we accepted other principles of arithmetic. But students were reluctant to do so because they knew that in Unit 1 we

(continued on T. C. 40F)

For every △, □, ◇, and ○,

△ (□ + ◇ + ○)

= △ [(□ + ◇) + ○] [convention; see T. C. 43C, D of Unit 1]

= △ (□ + ◇) + △ ○ [distributivity]

= [△ □ + △ ◇] + △ ○ [distributivity]

= △ □ + △ ◇ + △ ○ [convention]

*

Here is one way of handling Exercise 27.

4(9 - 3)	= 4[9 + (-3)]	[fact of arithmetic of directed numbers]
	= 4(9) + 4(-3)	[distributivity]
	= 4(9) + (-12)	[fact of arithmetic of directed numbers]
	= 4(9) - (12)	[fact of arithmetic of directed numbers]
	= 4(9) - 4(3)	[fact of arithmetic]

The phrase 'fact of arithmetic of directed numbers' requires explanation. When students learned in Unit 1 how to operate with directed numbers, they were actually learning many facts ['combinations' is a better word] of the arithmetic of directed numbers, just as in grade school they learned many facts of the arithmetic of the numbers of arithmetic. ['numbers of arithmetic' means the same thing as 'numbers of grade school'.] Just as they would say that the fact that $5 \times 9 = 45$ and the fact that $\frac{2}{3} \div \frac{7}{8} = \frac{2}{3} \times \frac{8}{7}$ are facts of arithmetic, so they should say that the fact that $4(-3) = -12$ and the

(continued on T. C. 40E)

*

Exercise 18 [Some students had trouble with this one.]

$$1(3 + 9) = 1(3) + 1(9) \qquad \text{[distributivity]}$$
$$= 1(3) + 9 \qquad \text{[extended principle of 1]}$$

Note that Exercise 18 is a true instance (and Exercise 17 a false instance) of a false generalization. Ask students to state this false generalization. Similarly, Exercise 19 is a false instance and Exercise 20 is a true instance of a generalization. Again, ask students to state the generalization in question.

*

Exercise 22

$$(3 + 5)6 + (3 + 5)7 = [3(6) + 5(6)] + [3(7) + 5(7)] \qquad \text{[second distributive principle]}$$

Note the tie between Exercises 21 and 22. From them we can derive:

$$(3 + 5)(6 + 7) = [3(6) + 3(7)] + [5(6) + 5(7)].$$

Ask students to state a true generalization of which the foregoing is an instance. Help them, if necessary, by giving a beginning for such a generalization:

For every ○ , △ , □ , and ⬡ ,

(○ + △)(□ + ⬡) = ...

Exercises 23, 24, and 25 provide the basis for extending the distributive principle.

(continued on T. C. 40D)

Exercise 14--Short way

$(2 + 0) + 8 = 2 + 8$ [principle of 0 for addition]

Long way

$(2 + 0) + 8 = 2 + (0 + 8)$ [associativity of addition]

$= 2 + (8 + 0)$ [commutativity of addition]

$= 2 + 8$ [principle of 0 for addition]

A student with a mental set engendered by Exercise 13 [and by the wicked Hint] might use the long way for Exercise 14. If this happens, urge him to seek a shorter way but assure him that his long way is correct.

*

Exercise 15

$3(5 + 9) = (5 + 9)3$ [commutativity of multiplication]

$= (9 + 5)3$ [commutativity of addition]

*

Exercise 16

$3(5 + 9) = 3(5) + 3(9)$ [distributivity]

$= 5(3) + 9(3)$ [commutativity of multiplication]

Exercise 16 may suggest to some students the principle:

For every △, □, and ◇,

$(\square + \diamond)\triangle = \square\triangle + \diamond\triangle$.

This principle is a very useful one [combining terms] and should be derived and discussed the first time someone suggests it. [That someone may be you!] A short name for this principle is 'the right distributive principle'.

(continued on T. C. 40C)

Exercise 13

$$
\begin{aligned}
(5 \times 0) \times 7 &= 5 \times (0 \times 7) && \text{[associativity of multiplication]} \\
&= 5 \times (7 \times 0) && \text{[commutativity of multiplication]} \\
&= 5 \times 0 && \text{[principle of 0 for multiplication]}
\end{aligned}
$$

In Exercise 13, a student may claim that '$0 \times 7 = 0$' is an instance of the principle of 0 for multiplication. This is incorrect because the principle of 0 for multiplication states that, for every $\square$, $\square \times 0 = 0$, and one could never obtain a replacement for '$\square$' which would produce the instance '$0 \times 7 = 0$'. The student has in mind, perhaps, the principle: For every $\square$, $0 \times \square = 0$, which can be derived from the principle of 0 for multiplication with the help of the commutative principle for multiplication.

For every $\square$,

$$
\begin{aligned}
0 \times \square &= \square \times 0 && \text{[case of commutative principle for multiplication]} \\
&= 0. && \text{[principle of 0 for multiplication]}
\end{aligned}
$$

[This derived principle is one which will be used frequently in the course. Because it is convenient to have short names for principles, you may want to call it 'the extended principle of 0 for multiplication'. Undoubtedly, a student will suggest "the extended principle of 0 for addition". When this happens, ask the student to state it, and then to derive it.]

*

(continued on T. C. 40B)

13. $(5 \times 0) \times 7 = 5 \times 0$ 14. $(2 + 0) + 8 = 2 + 8$

(Hint: You may have to use several principles in Exercises 13 and 14.)

15. $3(5 + 9) = (9 + 5)3$ 16. $3(5 + 9) = 5(3) + 9(3)$

17. $3(5 + 9) = 3(5) + 9$ 18. $1(3 + 9) = 1(3) + 9$

19. $3(5 + 9) = (3 + 5)9$ 20. $2(6 + 2) = (2 + 6)2$

21. $(3 + 5)(6 + 7) = (3 + 5)6 + (3 + 5)7$

22. $(3 + 5)6 + (3 + 5)7 = [3(6) + 5(6)] + [3(7) + 5(7)]$

23. $5[(9 + 1) + 3] = 5(9 + 1) + 5(3)$

24. $5(9 + 1 + 3) = [5(9) + 5(1)] + 5(3)$

25. $5(9 + 1 + 3) = 5(9) + 5(1) + 5(3)$

26. $4[9 + (-3)] = 4(9) + 4(-3)$

27. $4(9 - 3) = 4(9) - 4(3)$

28. $28 - 14 = 14(2 - 1)$

29. $5 + 3(4) = (5 + 3)(5 + 4)$

☆30. $\frac{1}{2} + \frac{1}{3}(\frac{1}{6}) = (\frac{1}{2} + \frac{1}{3})(\frac{1}{2} + \frac{1}{6})$

B. Some of the following statements are true and some are false. Give one or more principles which justify the true ones. For those which are false give a false instance.

Sample 1. If in the expression

'3 ⬡ + 5 ⬡ = 8 ⬡ '

we write in each '⬡' a numeral for any number, the resulting statement is true.

Solution. The given statement tells you that all of the instances of the following are true:

For every ⬡ , 3 ⬡ + 5 ⬡ = 8 ⬡ .

Each instance is a true statement because it is a consequence of the commutative and distributive principles.

In '⬡ (△ + ○) = ⬡ △ + ⬡ ○ '

of the distributive principle, replace '△' by '3' and '○' by '5' to obtain:

⬡ (3 + 5) = ⬡ 3 + ⬡ 5.

Apply the commutative principle for multiplication to obtain:

8 ⬡ = 3 ⬡ + 5 ⬡ .

Since each instance of 'For every ⬡ , 3 ⬡ + 5 ⬡ = 8 ⬡ ' is true, the generalization is true.

Sample 2. If in the expression

'2 ○ + 5 △ = 7 ○ △ '

we write in each '○' a numeral for any number, and we write in each '△' a numeral for any number, then the resulting statement is true.

Solution. The given statement states that each instance of the following is true:

For every ○ and △ , 2 ○ + 5 △ = 7 ○ △ .

ving just
hat

Exercise 1. The principle of 1, the commutative principle for multiplication, and the distributive principle.

Exercise 2. The commutative and associative principles for multiplication.

Exercise 3. False. Try '7' for ' $\square$ '. [Ask students to find a true replacement instance of '3 $\square$ + 5 = 8 $\square$ '.]

Exercise 4. False. Try '2' for ' $\triangle$ ' and '3' for ' $\square$ '. [Let students discover the fact that the pairs of replacements which give true replacement instances are pairs of numerals for the same number.]

Exercise 5. Commutative principle for multiplication, and the principle of distributivity for subtraction.

Exercise 6. [As in Exercise 2.]

We can prove that the given statement is false by giving just one counter-example, that is, a pair of numbers such that when their names replace '◯' and '△', we get a false instance.

Try '3' for '◯' and '4' for '△':

$$2(3) + 5(4) = 7(3)(4)$$

$$2(3) + 5(4) = 26$$

$$7(3)(4) = 84$$

$$26 \neq 84$$

1. If in the expression '$\bigcirc + 4\bigcirc = 5\bigcirc$' we write in each '$\bigcirc$' a numeral for any number, the resulting statement is true.

2. If in the expression '$2\triangle \times 3\bigcirc = 6\triangle\bigcirc$' we write in each '$\triangle$' a numeral for any number, and we write in each '$\bigcirc$' a numeral for any number, the resulting statement is true.

3. If in the expression '$3\square + 5 = 8\square$' we write in each '$\square$' a numeral for any number, the resulting statement is true.

4. If in the expression '$5\triangle + 6\square = 6\triangle + 5\square$' we write in each '$\triangle$' a numeral for any number, and we write in each '$\square$' a numeral for any number, then the resulting statement is true.

5. If in the expression '$8\bigcirc - 2\bigcirc = 6\bigcirc$' we write in each '$\bigcirc$' a numeral for any number, then the resulting statement is true.

6. If in the expression '$4\triangle \times (-2)\bigcirc = -8\triangle\bigcirc$' we write in each '$\triangle$' a numeral for any number, and we write in each '$\bigcirc$' a numeral for any number, then the resulting statement is true.

Part B (cont.)

Exercise 7. Distributive principle, and the associative principle for multiplication.

Exercise 8. Commutative principle for multiplication, and the principle of distributivity for subtraction.

* * *

An important transition. By now, the student ought to understand that a pronumeral stands neither for a number nor for a numeral. The pronumeral stands in place of, or holds a place for, a numeral. You may even want to tell the students to think of 'x' as ' ' so that a numeral can be written in it, for example:

.

7. If in the expression '$\frac{1}{8}(4\ \bigcirc\ +\ 2\ \bigcirc\ ) = \frac{1}{2}\ \bigcirc\ +\frac{1}{4}\ \bigcirc$ ' we write in each '$\bigcirc$' a numeral for any number, and we write in each '$\bigcirc$' a numeral for any number, then the resulting statement is true.

8. If in the expression '$5\Delta\ \bigcirc\ \diamond\ - 2\ \bigcirc\ \diamond\ =\ \bigcirc\ \diamond\ (5\Delta - 2)$' we write in each '$\Delta$' a numeral for any number, and we write in each '$\bigcirc$' a numeral for any number, and we write in each '$\diamond$' a numeral for any number, then the resulting statement is true.

C. Simplify the following expressions. Use the principles of arithmetic to shorten computations. Do as much as you can mentally.

1. $6(\frac{1}{2} + \frac{1}{3})$
2. $25 \times 40 \times \frac{1}{5}$
3. $5 \times 97 \times 2$
4. $7 + 26 + 3 + 44$
5. $17(19) + 17(-19)$
6. 43×99
7. 99×101
8. $75 + (-97) + (-75) + (97)$
9. $69 \times \frac{1}{2} + 11 \times \frac{1}{2}$
10. $7\frac{1}{7} \times 999 \times 7$

2.06 Pronumerals and letter symbols.--No doubt you have become tired of writing pronumerals like '$\bigcirc$', '$\square$', 'Δ', '$\bigcirc$', and '$\diamond$'. The reason we have used symbols like these is to emphasize the fact that these symbols act like frames in which numerals are to be written. After a numeral is written, it does not matter whether you keep the frame or not. By now you should understand that a pronumeral is not a symbol for a number. As long as this fact is clear to you, we can make the job of writing expressions containing pronumerals much easier by using letter symbols as pronumerals.

For example, consider the statement of the commutative principle for addition:

For every and , + = + .

Instead of ' ' we can write 'x', and instead of ' ', we can write 'y'. Then the commutative principle for addition is stated:

For every x and y, $x + y = y + x$.

Any letter symbols can be used for pronumerals. We can use, for example, 'k' instead of 'x' and 'R' instead of 'y':

For every k and R, $k + R = R + k$.

EXERCISES

A. Each of the following expressions contains a pronumeral. In each exercise, choose any number you please. Write a numeral for this number in place of the pronumeral in the exercise. Tell whether the resulting statement is true or false. Make your choices so that some of the statements are true and some are false.

Sample. $3x - 2 = 7x + 4$

Solution. Substitute for 'x' a numeral for any number. For example, substitute '5' for 'x':

$$3(5) - 2 = 7(5) + 4.$$

Is this statement true or is it false?

$$3(5) - 2 = 15 - 2 = 13$$

$$7(5) + 4 = 35 + 4 = 39$$

$$13 \neq 39$$

Therefore, the statement is false.

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Is this statement true or is it false?

$$3(5) - 2 = 15 - 2 = 13$$

$$7(5) + 4 = 35 + 4 = 39$$

$$13 \neq 39$$

Therefore, the statement is false.

Use the words 'sentence' and 'equation' in referring to the equations in Part A. Avoid 'expression' here, except when referring to a side of an equation.

* * *

The students are not "solving" these equations. Caution them that after replacement, some of their statements should be true and some false. Of course, when they obtain a true statement, they have solved the equation, but avoid this terminology here. These matters will come up in Unit 3. However, you can keep your able students busy and happy by challenging them to find true replacement instances. [Do not give this kind of "extra" assignment as homework because the urge to get help from home is too strong.] Also, you may want to try Mrs. Catlow's trick of asking students to find a number which will give a false replacement instance in such exercises as 7, 16, 19, and 20. She reported that very few of her students fell for this one.

* * *

In Exs. 17-20 do <u>not</u> introduce exponent notation. If some students already know how to use this notation, let them do so on their own papers. We do not bring in exponents until much later in the course. The student's symbolism load is heavy enough without introducing another symbol which is frequently the cause of many mechanical errors. We think a student is less likely to confuse 'x^2' with '2x' if he works with 'xx' for a while before using exponent symbols.

1. $5x + 4 = 14$
2. $5 - 8y = -3$
3. $A + 4 = 5 + 6A$
4. $7t - 1 = 1 + t$
5. $3z = z + 8$
6. $10 - 2s = s + 5$
7. $2B + 4 = 2(2 + B)$
8. $8 - 3c = 1\frac{1}{2}(5\frac{1}{3} - 2c)$
9. $x + 1 = x$
10. $2z - 1 = 1 + 2z$
11. $\frac{5r - 1}{5r + 1} = 1$
12. $\frac{3m + 2}{8 - 3m} = \frac{6m + 5}{2 - 4m}$
13. $6j - j = 2$
14. $k - 3k = 8$
15. $10g - 8\frac{1}{2}g = 1.5g$
16. $12f + 7\frac{1}{3}f = 19\frac{1}{3}f$
17. $3xx - 12 = 0$
18. $uu = u + 20$
19. $(v - 7)(v + 7) = vv - 49$
20. $(w - 3)(2 + w) = ww - w - 6$

B. Each of the following exercises has several parts. Make the substitution indicated in each part and tell whether the resulting statement is true or false.

1. $3x - 2y = 7x + y$
   (a) '2' for 'x' and '3' for 'y'
   (b) '-3' for 'x' and '4' for 'y'
   (c) '3' for 'x' and '-4' for 'y'

2. $7t - 5s = 5t - 7s$
   (a) '0' for 't' and '0' for 's'
   (b) '$-\frac{1}{2}$' for 't' and '$\frac{1}{2}$' for 's'
   (c) '7' for 't' and '-6' for 's'

The first says that the product of each number by its reciprocal is $^{+}1$, so that, for instance, $^{+}2 \cdot (/ + 2) = {}^{+}1$. The second, on the other hand (if ' $-$ ' is taken as referring to division), is a consequence of the definition of division as the inverse of multiplication. Moreover, the first is needed in proving (and even in stating)

The Principle of Division

For each x and $y \neq 0$, $x/y = x \cdot (/y)$.

In practice the principle of division is stated:

For each x and $y \neq 0$, $\frac{x}{y} = x \cdot \frac{1}{y}$

and here, as in part 1 of the statement of the principle of reciprocals, '1' is interpreted as the reciprocation operator. But in other contexts '1' is a name for the directed number $^{+}1$, ' $-$ ' is a name for the operation of division, and '1' by itself refers to nothing [just as in '$\frac{2}{3}$', neither of the parts '2' and '$\frac{}{3}$' has a meaning].

* * *

At last, rules for operating with directed numbers! We are sure that by this time, the statements of the rules are really unnecessary for correct performance with directed numbers. The student should treat the statements as reading exercises. It is important for him to learn that his intuitive ideas for arithmetic with directed numbers can be spelled out carefully as rules, even though he doesn't formally use these rules in doing the arithmetic.

Comparison of the above with the principle of opposites shows up an inadequacy in the notation used in talking about directed numbers. There is an operator ' – ' which is used in naming the opposite of a directed number (see part 1 of the principle of opposites), but there is no operator in common use for forming names of reciprocals of directed numbers. A natural choice for such an operator is '/'. Using this, '/ + 2' names the reciprocal of $^{+}2$, and the two parts of the principle of reciprocals should be written:

For every $x \neq 0$, $x \cdot (/x) = {}^{+}1$

and:

For every x and y, if $x \cdot y = {}^{+}1$ then $y = /x$.

That such a notation as ' – ' and '/' are desirable becomes evident whenever one tries to explain the difference between the statement made in part 1 of the principle of opposites:

For every x, $x + (-x) = 0$

and in the sentence:

For every x, $x + (0 - x) = 0$.

The first, as previously noted, states that the sum of each number with its opposite is 0, so that, for instance, $^{+}2 + (-{}^{+}2) = 0$. The second, on the other hand, is a consequence of the definition of subtraction as the inverse of addition. Note also that one uses the first in proving (see T. C. 40G)

The Principle of Subtraction

For each x and y, $x - y = x + (-y)$.

There should be a similar difference between:

For every $x \neq 0$, $x \cdot (/x) = {}^{+}1$

and:

For every $x \neq 0$, $x \cdot \frac{{}^{+}1}{x} = {}^{+}1$.

(continued on T. C. 46D)

Non-recognition of the difference between usages (i) and (ii) can cause confusion. One gimmick you can use in explaining this distinction consists in using symbols like '$^{+}3$' and '$^{-}2$' as numerals for directed numbers and use the symbol ' $-$ ' only according to usages (ii) and (iii). For example, that $^{+}2$ is the opposite of $^{-}2$ can be expressed by '$^{+}2 = -(^{-}2)$'; and that every number is the opposite of its opposite can be expressed by 'for every x, $-(-x) = x$'. With this notation, '$^{-}(^{+}2)$' is nonsense [since '$^{-}$' should be prefixed only to numerals for numbers of arithmetic], and '-2' is nonsense, unless '2' is being used as an abbreviation for '$^{+}2$', [since ' $-$ ' should be prefixed only to numerals for directed numbers]. For symmetry, the only usages of the symbol '+' will be analogous to (ii) and (iii) [its function in constructing numerals for positive directed numbers being taken over by '$^{+}$']:

(ii') by prefixing '+' to a numeral for a directed number we obtain a numeral for the same directed number, and

(iii') by inserting it between two numerals for directed numbers [or between two numerals for numbers of arithmetic] we obtain a numeral for the sum of these numbers. You may find this gimmick about the superscript signs, '$^{+}$' and '$^{-}$', helpful in explaining the materials on pages 2-47 and 2-48.

* * *

The Principle of Reciprocals

1. For every $x \neq 0$, $x \cdot \frac{1}{x} = 1$.

2. For every x and y, if $x \cdot y = 1$ then $y = \frac{1}{x}$.

(continued on T. C. 46C)

Students should insert in their textbooks (correct) statements of the principles listed in Part C.

* * *

If, in discussing Unit 1, you talked about a principle of opposites and a principle of reciprocals, you may want to list these along with the others in Part C. You will need to spend some time in identifying these principles [See T. C. 40H]. The first is:

The Principle of Opposites

1. For every x, $x + (-x) = 0$.
2. For every x and y, if $x + y = 0$ then $y = -x$.

Less formally, part 1 of the principle states that the sum of each number with its opposite is 0, while part 2 states that if the sum of two numbers is 0 then the second is the opposite of the first.

[In Unit 1, the word 'opposite' was explained by saying that the opposite of a positive number is the "corresponding" negative number; the opposite of a negative number is the "corresponding" positive number; and the directed number 0 is its own opposite.]

The symbol '—' is used in three ways:

(i) by prefixing it to a numeral for a number of arithmetic we obtain a numeral for a negative directed number (or for the directed number, zero),

(ii) by prefixing it to a numeral for a directed number we obtain a numeral for the opposite directed number, and

(iii) by inserting it between two numerals for directed numbers we obtain a numeral for the difference of the second number from the first.

(continued on T. C. 46B)

3. $3abc - 2a = 4b - 5c$

   (a) '5' for 'a', '1' for 'b', and '.7' for 'c'
   (b) '-1' for 'a', '+2' for 'b', and '-6' for 'c'
   (c) '$1\frac{1}{2}$' for 'a', '$\frac{3}{5}$' for 'b', and '$\frac{2}{3}$' for 'c'

4. $\frac{3u - 2v}{5u + 3v} = \frac{2}{5}$

   (a) '16' for 'u' and '5' for 'v'
   (b) '3' for 'u' and '1' for 'v'
   (c) '-3.2' for 'u' and '-1' for 'v'

5. $(a + b)(2a - 3b) = 2aa - ab - 3bb$

   (a) '-2' for 'a' and '3' for 'b'
   (b) '5' for 'a' and '-4' for 'b'
   (c) '$\frac{3}{4}$' for 'a' and '$\frac{2}{3}$' for 'b'

C. Use letter symbols for pronumerals and state the following principles of arithmetic:

1. The principle of one
2. The principle of zero
3. The commutative principle for addition
4. The commutative principle for multiplication
5. The associative principle for addition
6. The associative principle for multiplication
7. The distributive principle

2.07 Pronumerals and directed numbers.--In Unit 1 you learned how to add, subtract, multiply and divide directed numbers, and you developed your own procedures for carrying out these operations. We did not state any rules in that unit because it was better for you to discover rules for yourself. Now we can state such rules in a compact fashion by using pronumerals.

OPPOSITE NUMBERS

You know how to find the opposite of a given number. For example, to find the opposite of 9, you find the number which when added to 9 gives the sum 0. This number is -9 because 9 + (-9) = 0. Now, in all likelihood, you do not follow this procedure in finding the opposite of, say, 9. Most students (and mathematicians, too) use a more mechanical procedure. They simply write a numeral for 9 and place a ' - ' at the left of the numeral. The resulting symbol is itself a numeral; it stands for the number -9.

Now, let us agree to use the same mechanical procedure to find the opposite of a negative number, say, -9. Then '--9' is a name for the opposite of -9. Instead of writing '--9' which is awkward looking, we usually write '-(-9)'. Then, we have:

-(-9) is the opposite of -9.

Since we already know that +9 is the opposite of -9, we have:

$$-(-9) = +9.$$

What is the opposite of -(-9)? Using the same mechanical procedure as before, we can say that

-[-(-9)] is the opposite of -(-9).

Since -(-9) is +9, the opposite of -(-9) is -9. Therefore,

$$-[-(-9)] = -9.$$

What is the opposite of 3? Of -3? Of -(-3)? Of -[-(-3)]?

We can use this mechanical procedure to write a name for the opposite of any given number. Suppose we want to find the opposite of the number 3 - 7. The name of its opposite is '-(3 - 7)' [Why do we use parentheses?]. But 3 - 7 is -4, and the opposite of -4 is +4. Hence,

$$-(3 - 7) = +4.$$

Let us summarize all of the cases we have treated above and provide for all of the cases we might ever consider by using pronumerals to state the following principle.

For every x, -x is the opposite of x.

Sometimes you will come across an expression such as:

$$+(-6)$$

You can probably guess that this expression stands for the same number as does:

$$-6$$

In other words $+(-6) = -6$. Study the following examples:

$$+5 = 5$$
$$+(-4) = -4$$
$$+[-4 + (-9)] = -4 + (-9)$$

We can summarize this simplification procedure as follows:

For every x, +x = x.

EXERCISES

A. Find the opposite of each of the numbers listed in the following exercises, and give its simplest name.

Sample: $-(-3)$

Solution: Using the rule above we know that $-[-(-3)]$ is the opposite of $-(-3)$. We can simplify '$-[-(-3)]$' to get '-3'.

1. $+2$	2. -7	3. $-(-8)$	4. $-(-0)$
5. $-(+1)$	6. $-(-1)$	7. $-[-(-5)]$	8. $-[-(+1)]$

Sample: $334.6 - \frac{22}{57}$

Solution: The opposite of the given number is

$$-(334.6 - \frac{22}{57}).$$

(continued on next page)

We could simplify this expression further but we will not because we are concerned mainly with learning to apply the rule for finding opposites. In the following exercises give the opposite but do not simplify.

9. $\frac{3}{5}$ 10. $-\frac{7}{10}$ 11. $\frac{8+4}{7-12}$ 12. $\frac{9 \times 856}{4 \times 593}$

13. $84,967\frac{15}{16} - 34,985\frac{27}{59} - 41,985.6 + 7(65.2)$

B. In Unit 1 we agreed that if two numbers were opposites, their sum would be 0. That is, we agreed that

For every x, $x + (-x) = 0$.

Check to see that this is the case by substituting several numerals for 'x' in the expression '$x + (-x) = 0$'.

C. For each of the following expressions write another such that the two expressions name opposite numbers whenever the pronumerals are replaced by numerals.

1. y	2. -z	3. -(-x)
4. 2a	5. -3b	6. -(-8c)
7. $\frac{1}{2}k$	8. -7j	9. 10(k + m)
10. -xy	11. xy	12. x + y
13. x - y	14. y - x	15. $\lvert x - y \rvert$

D. Check each of the following statements.

Sample. For every x, -x is a negative number.

Solution. Recall that the phrase 'for every x' means that 'x' can be replaced by numerals for positive numbers, for negative numbers, and for the number 0. Now, when a numeral is substituted for 'x' in the expression '-x is a negative number', we have an instance of the given general statement. We must determine whether this instance is true or false.

be true. Similarly, we want to use the words 'if ... then _ _ _' in such a way that each conditional sentence is a consequence of its contrapositive. For example, we want (2) to be a consequence of:

(4) if Boston is not in Massachusetts then $1 + 1 \neq 2$,

and we want (3) to be a consequence of:

(5) if Boston is not in Massachusetts then $7 + 8 \neq 3$.

So, beside the criterion for truth, each consequence of a true sentence is true, we have two criteria for consequence which can be symbolized by:

(i) $\dfrac{q}{p \Longrightarrow q}$, and: (ii) $\dfrac{\sim q \Longrightarrow \sim p}{p \Longrightarrow q}$.

[(i) is read as '$p \Longrightarrow q$ is a consequence of q'; (ii) is read as '$p \Longrightarrow q$ is a consequence of not $q \Longrightarrow$ not p'.] Let us apply these criteria to the instance generated from Exercise 2:

(6) if -3 is a positive number then -(-3) is a negative number.

This sentence, by (ii), is a consequence of:

(7) if -(-3) is not a negative number then -3 is not a positive number,

and (7), by (i), is a consequence of:

(8) -3 is not a positive number.

Hence, (6) is a consequence of (8). Since we certainly want to label (8) 'true', we must correspondingly label (6) 'true'.

Students who have mastered the procedure for generating instances of a generalization may wonder about the following instance of Exercise 2:

if -3 is a positive number then
-(-3) is a negative number.

There should be no question about whether this instance is a consequence of the generalization. The fact that it is a consequence is merely what we mean by 'For every x' [See T. C. 35E]. Now, you cannot give the student at this time a complete logical analysis of the reasons for labelling this instance 'true'. However, you may get somewhere by getting the students to agree that they want to label the generalization 'true', and since we have decided that consequences of true sentences are to be labelled 'true', we must label this instance 'true'.

Of course, the real reasons have to do with the conditional form of the sentence and not with the fact that it is an instance of a generalization. Briefly, we want to use the words 'if ... then ___' in such a way that every conditional sentence follows from its consequent (that is, from its "then part"). And we want to use the word 'true' in such a way that consequences of true sentences are true. If both these wants are to be satisfied then we must label 'true' each conditional sentence whose consequent is true. For example, suppose we label 'true' the sentence:

(1) Boston is in Massachusetts.

Then, we shall label 'true' the following conditional sentences:

(2) if 1 + 1 = 2 then Boston is in Massachusetts,

and:

(3) if 7 + 8 = 3 then Boston is in Massachusetts,

because we want each of these conditional sentences to be a consequence of its consequent, (1); and we want consequences of true sentences to

(continued on T. C. 50B)

Let us substitute '5' for 'x':

-(5) is a negative number.

The statement '-(5) is a negative number'. is true. Now we stubstitue for 'x' a numeral for a negative number, say, -6 and obtain:

-(-6) is a negative number.

But we know

$$-(-6) = +6.$$

Therefore, the statement '-(-6) is a negative number' is a false statement.

Hence, we know that the given statement:

For every x, -x is a negative number

is false because we have found a counter-example, a number which leads to a false instance.

1. For every x, +x is a positive number.
2. For every x, if x is a positive number then -x is a negative number.
3. For every x, if x is a negative number then -x is a positive number.
4. For every x, if x is 0 then -x is 0.
5. For every x, if -x is a positive number then x is a negative number.
6. For every x, if -x is a negative number then x is a positive number.

ABSOLUTE VALUE

In Unit 1 you became familiar with the idea of the absolute value of a number. For example, you learned that the absolute value of -6 was +6, and that the absolute value of +3 was +3. But in Unit 1 you did not state a definition which could be applied to every number. Just as in the case of the principles of arithmetic, it would be impossible to write down a table listing the absolute value of every number. So, we use pronumerals. If someone asked you how to find the absolute value of a number, you might reply:

In line with the discussion on T. C. 64A of Unit 1, we point out that although the boxed statement on absolute value is the one found in most analysis texts, we ourselves have not completely settled the matter of deciding whether it is more convenient to define the absolute value of a directed number as a directed number or as a number of arithmetic.

* * *

In Part A you will want to use some of the items to teach students how to use the formal statement of the rule in the box. Consider Exercise 1. Since $-7 < 0$, the student should use (b) of the rule. In this case, then, $|-7|$ is the opposite of -7. That is, $|-7| = +7$.

In Exercise 4 we have a case where the student must find another name for the number in question. Instead of '5 - 2', the student needs to use the name '3'. Then he applies (a) of the rule.

In Exercise 3 the student must use the rule twice. He is told to find the absolute value of an absolute value. First, he uses the rule to find a name simpler than '$|-6|$'. Part (b) of the rule tells him that another name for $|-6|$ is '+6'. He has now changed his problem to:

Find the absolute value of +6,

and (a) of the rule tells him $|+6| = +6$.

* * *

Mr. Marston suggests a fifth exercise for Part C on page 2-52:
For every x, $|(x)(-x)| = xx$.

"If the number is positive or zero, it is equal to its absolute value, and if the number is negative, its absolute value is its opposite."

Notice that the word 'it' is playing the part of a pronumeral in the statement above. Using the pronumeral 'x' the statement becomes:

For every x,
(a) if $x \geq 0$ then $|x| = x$;
(b) if $x < 0$ then $|x| = -x$.

Let us apply this statement to several examples.

Example 1. What is the absolute value of +5?

Solution. Since $+5 \geq 0$, we use part (a) of the statement. The statement tells us that

$$|+5| = +5.$$

Example 2. What is the absolute value of -6?

Solution. Since $-6 < 0$, we use part (b) of the statement:

$$|-6| = -(-6) = +6.$$

Example 3. What is the absolute value of 0?

Solution. Since $0 \geq 0$, we use part (a):

$$|0| = 0.$$

EXERCISES

A. Find the absolute value.

1. -7
2. $3\frac{1}{2}$
3. $|-6|$
4. $5 - 2$
5. $2 - 5$
6. $-(5 + 7)$
7. $5 + 7$
8. $-(-3)$
9. $7 - 7$

B. Check each of the following statements.

1. For every x, $|-x| = -x$.
2. For every x, $|-x| = x$.
3. For every x, $|x| = x$.
4. For every x, $|x| = -x$.
5. For every x, $|-x| = |x|$.

C. The following statements are true. Check them.

1. For every x, $|x| \geq x$.
2. For every x, $|xx| = xx$.
3. For every x, $-|xx| = x(-x)$.
4. For every x except 0, $\left|\frac{x}{x}\right| = \frac{x}{x}$.

COMPARING DIRECTED NUMBERS

If you are given two directed numbers, say, -7 and +3, you can tell which number is the greater by subtracting one number from the other.

$$-7 - (+3) = -10$$

Since -10 is a negative number, then we know that

$$-7 < +3.$$

If we subtract -7 from +3, we obtain:

$$+3 - (-7) = +10.$$

Since +10 is a positive number, then we know that

$$+3 > -7.$$

Of course, the two expressions

'$+3 > -7$' and '$-7 < +3$'

tell exactly the same thing.

We can state the procedure for comparing any two numbers as follows:

> For every x and y,
> (a) if x - y is a positive number then $x > y$;
> (b) if x - y is a negative number then $x < y$;
> (c) if $x - y = 0$ then $x = y$.

We could formulate a more elegant statement if we had a brief notation for referring to the number of arithmetic corresponding with a directed number, and a brief notation for referring to the directed number corresponding with a number of arithmetic. One of the conveniences of defining the absolute value of a directed number to be a number of arithmetic is that such a brief notation is then available. For example, '$|+7|$' is a symbol for the number of arithmetic which corresponds with +7, and '+(7)' is a symbol for the directed number which corresponds with the arithmetic number 7. The rule for addition could then be stated:

For every x and y,
if $x \geq 0$ and $y \geq 0$
then $x + y = +(|x| + |y|)$.

However, we cannot use this statement because we have defined the absolute value of a directed number to be a directed number.

* * *

Try to get students to state the rule for adding negative numbers before they turn to page 2-54. Start them on it by writing:

For every x and y,
if $x < 0$ and $y < 0$ then $x + y = \ldots$.

Ask them to examine carefully what they do when they carry out the examples at the bottom of page 2-53. Ask if it would be possible for someone to use his knowledge of adding positive numbers in trying to add negative numbers. How do you get from a negative number to the corresponding (opposite) positive number? [Someone may suggest multiplying by -1. Point out that we are pretending at this time that we don't know how to multiply directed numbers.]

In general, try to evoke from the students all of the operation rules stated on 2-54, 2-57, 2-60, 2-62, and 2-63.

We urge that you add the following sentences to those in Part B.

10. For every x and y, if $x > y$ then there is a positive number z such that $x = y + z$.
11. For every x and y, if there is a positive number z such that $x = y + z$ then $x > y$.
12. For every x, y, and z, if $x \geq y$ then $x + z \geq y + z$.
13. For every x, y, and z, if $x + z \geq y + z$ then $x \geq y$.

Principles 10 and 11 serve as a definition of the relation $>$, and are useful in deriving properties of $>$ such as those expressed in 12 and 13.

In Unit 3 we shall ask you to refer to generalizations 12 and 13 in formulating transformation principles for inequations. At this time, students need only verify 12 and 13. They should also notice that each instance of 13 is equivalent to an instance of 12, and conversely. For example, an instance of 13 is:

$$\text{if } 7 + 4 \geq 2 + 4 \text{ then } 7 \geq 2,$$

and the corresponding instance of 12 is:

$$\text{if } 11 \geq 6 \text{ then } 11 + (-4) \geq 6 + (-4).$$

Thus, by the principle of subtraction (of directed numbers), each of generalizations 12 and 13 is derivable from the other.

* * *

Note that we do not state a rule for finding the sum of positive numbers. Such a rule might be:

> The sum of a positive number and a positive number is the positive number corresponding to the number of arithmetic which is the sum of the numbers of arithmetic corresponding to the given addends.

(continued on T. C. 53C)

In Part A the students are to practice using the rule stated at the bottom of page 2-52. It is hardly a challenge at this point just to give the correct answer.

* * *

For a discussion as to why such instances of the generalization in Exercise 1 as:

(1) if 3 - 2 is a positive number then 2 - 3 is a negative number,

(2) if 2 - 3 is a positive number then 3 - 2 is a negative number,

and

(3) if 2 - 2 is a positive number then 2 - 2 is a negative number,

are true, see T. C. 50A, B. Briefly, (1) is labelled 'true' because it is a consequence of the true sentence '2 - 3 is a negative number'; (2) is labelled 'true' because it is a consequence of its contrapositive:

if 3 - 2 is not a negative number then 2 - 3 is not a positive number,

which is true because it is a consequence of the true sentence '2 - 3 is not a positive number'; the reason for labelling (3) 'true' is similar to that just given in the case of (2).

* * *

Exercises 1, 2, and 3 of Part B suggested to a University High School student that, for every x and y, x - y is the opposite of y - x. Here is how he derived this generalization.

$$
\begin{aligned}
(x - y) + (y - x) &= [x + (-y)] + [y + (-x)] \qquad \text{[See page 2-60.]} \\
&= [x + (-x)] + [y + (-y)] \\
&= \quad 0 \quad + \quad 0 \\
&= 0
\end{aligned}
$$

Therefore, by the principle of opposites, x - y is the opposite of y - x.

* * *

(continued on T. C. 53B)

EXERCISES

A. Tell which number is larger.

1. +3, +8	2. -5, -6	3. -10, +12
4. +10, -12	5. 8, 2	6. +8, 2
7. -9, 9	8. 0, -2	9. 0, +2

B. Check these statements.

1. For every x and y, if x - y is a positive number then y - x is a negative number.
2. For every x and y, if x - y is a negative number then y - x is a positive number.
3. For every x and y, if x - y is 0 then y - x is 0.
4. For every x, $x + 2 > x$.
5. For every x, $x - 2 < x$.
6. For every x, $2x > x$.
7. For every x, $\frac{x}{2} < x$.
8. For every x, y, and z, if $x < y$ and $y < z$ then $x < z$.
9. For every x, y, and z, if $x < y$ and $x < z$ then $y < z$.

ADDITION

In Unit 1 you learned how to add by interpreting an addition problem as a combining of trips. You probably developed your own rules for adding directed numbers but you were not required to state them. Now, you can easily state these rules by using pronumerals.

Sum of two positive numbers

This case was very easy because you compared positive directed numbers with the numbers of arithmetic. In fact, you need only remember that positive numbers act like the numbers of arithmetic when you add them.

Sum of two negative numbers

Add -6 and -3: $-6 + (-3) = -9$

Add -8 and -2: $-8 + (-2) = -10$

Add $-3\frac{1}{2}$ and $-5\frac{1}{4}$: $-3\frac{1}{2} + (-5\frac{1}{4}) = -8\frac{3}{4}$

Read the following statement. Does it express what you do to find the sum of two negative numbers?

> For every x and y,
>
> if $x < 0$ and $y < 0$ then $x + y = -(|x| + |y|)$.

Sum of a positive number and a negative number

Add +3 and -6:

$+3 + (-6) = -3$

Add -3 and +6:

$-3 + (+6) = +3$

Add $+\frac{1}{2}$ and $-\frac{3}{4}$:

$+\frac{1}{2} + (-\frac{3}{4}) = -\frac{1}{4}$

Add $-\frac{3}{8}$ and $+1\frac{3}{4}$:

$-\frac{3}{8} + (+1\frac{3}{4}) = +1\frac{3}{8}$

Does the following statement express what you do to find the sum of a positive number and a negative number?

> For every x and y,
>
> if $x < 0$ and $y > 0$,
>
> (a) and if $|x| < y$ then $x + y = y - |x|$;
>
> (b) but if $|x| > y$ then $x + y = -(|x| - y)$.

Note that if one or both of the numbers to be added is 0, then you find the sum by applying the principle of zero.

The boxed statements above merely summarize some of the procedures you may use to add directed numbers. It is pointless to memorize these statements since you already know how to add directed numbers. But you ought to be able to read and understand the statements. To help you understand the statements, you should try to apply them to specific cases.

In doing exercises in Part A the students are to use the boxed statements on page 2-54 and decide which one is applicable. For example, in doing Exercise 1, the student must first determine that the second boxed statement is the one to use. Next, he determines that the 'x' will be replaced by '-5' and the 'y' by '9'. Finally, he decides that part (a) should be used.

To assure that the students follow these steps, it may be wise to do the exercises of Part A orally.

Example 1. Add -7 and -5.

Solution. Since this problem asks for the sum of two negative numbers, use the statement in the first box and page 2-54. In the expression 'if $x < 0$ and $y < 0$ then $x + y = -(|x| + |y|)$' we substitute '-7' for 'x' and '-5' for 'y'. Now we have the statement 'if $-7 < 0$ and $-5 < 0$ then $-7 + (-5) = -(|-7| + |-5|)$'. But $|-7| = +7$ and $|-5| = +5$ and $+7 + (+5) = +12$. Since $-(+12) = -12$, we know that

$$-7 + (-5) = -12.$$

Example 2. Add -3 and +7.

Solution. We have a negative number and a positive number. So, we use the statement in the second box on page 2-54. Since $-3 < 0$ and $+7 > 0$, we replace 'x' by '-3' and 'y' by '+7'. Since $|-3| < +7$, we use part (a). Then, we substitute '-3' for 'x' and '+7' for 'y' in the expression

'$x + y = y - |x|$'

to obtain the true statement

'$-3 + (+7) = +7 - |-3|$'

But, $+7 - |-3| = 7 - 3 = 4$.

Therefore,

$$-3 + (+7) = 4.$$

EXERCISES

A. Use the procedure illustrated in the Examples to add the directed numbers listed in each exercise.

1. 9, -5	2. 3, -7	3. -2, +8
4. -3, -6	5. 2, -4	6. -7, +7

last point, for too many students seem to feel that it is proper to speak of such a generalization as being "sometimes true". A generalization is either true or false, and not a "mixture".]

2. True. If, in solving Exercise 1, students have described the true instances of that generalization, they need now only show that each instance of Exercise 2 which corresponds to a choice of two numbers, of which one is positive and the other negative, is true. But the sum of two such numbers is "between" the two numbers, so its absolute value is less than the larger of the absolute values of the two numbers.

3. False. It is easy to find false instances of this generalization. Another approach is to note that if the generalizations in Exercises 2 and 3 were both true then that of Exercise 1 would be true. Since this is not the case, and since the answer to Exercise 2 is 'true', the answer to Exercise 3 must be 'false'.

4. True.

5. False.

After working with the foregoing derivation, ask students to derive Exercise 4.

$$
\begin{aligned}
(x + y) + [(-x) + (-y)] &= [(x + y) + (-x)] + (-y) && \text{[associativity]} \\
&= \{x + [y + (-x)]\} + (-y) && \text{[associativity]} \\
&= \{x + [(-x) + y]\} + (-y) && \text{[commutativity]} \\
&= \{[x + (-x)] + y\} + (-y) && \text{[associativity]} \\
&= [x + (-x)] + [y + (-y)] && \text{[associativity]} \\
&= 0 + 0 && \text{[principle of opposites]} \\
&= 0 && \text{[principle of 0 for addition]}
\end{aligned}
$$

Since, for each x and y, $(x + y) + [(-x) + (-y)] = 0$, it follows from part 2 of the principle of opposites that $(-x) + (-y)$ is the opposite of $x + y$. [It also follows from the commutative principle for addition that $[(-x) + (-y)] + (x + y) = 0$, and so from part 2 of the principle of opposites that $x + y$ is the opposite of $-x + (-y)$ (Exercise 3).] It is important that students go through this derivation for it will provide them with a method of attack in deriving an analogous generalization about reciprocals which will be used in Part E on page 2-64.

* * *

Part C

1. False. It is instructive to ask students to describe the set of true instances of this false generalization. The generalization yields true instances when both numbers are positive, when both numbers are negative, and when either number is 0. Even though the generalization has many true instances, the generalization is false since it has at least one false instance. [Stress this

(continued on T. C. 56D)

of the principle of opposites it follows that

(c) $[-(-7) + (-5)] + [(-7) + 5] = 0$:

from this and the commutative principle for addition it follows that

(d) $[(-7) + 5] + [-(-7) + (-5)] = 0$,

and from this and part 2 of the principle of opposites it follows that

(a) $-(-7) + (-5) = -[(-7) + 5]$. [Vice-versa, from (a) and part 1 of the principle of opposites one can infer (d); from (d) and the commutative principle for addition one can infer (c); and from (c) and part 2 of the principle of opposites one can infer (b).]

5. For every two numbers, the absolute value of their sum is the sum of the absolute values of their opposites.

[Ask students why it is wrong to translate 4 as: The sum of every two negative numbers is the opposite of the sum of the corresponding positive numbers.]

Some students may be able to establish (as well as verify) Exercises 1 and 2. Here is how one student at University High School did this. Using the principle that, for every x, y, and z, if $x \geq y$ then $x + z \geq y + z$ [see T. C. 53B], we have the following derivation:

$$\begin{aligned} x + y &\geq 0 \\ (x + y) + [-x + (-y)] &\geq 0 + [-x + (-y)] \\ [x + (-x)] + [y + (-y)] &\geq 0 + [-x + (-y)] \\ 0 + 0 &\geq 0 + [-x + (-y)] \\ 0 &\geq -x + (-y) \end{aligned}$$

* * *

(continued on T. C. 56C)

Before students verify (or find counter-examples in the case of Exercise 5) the exercises in Part B, they should say aloud what each statement tells them. At this stage in their development, students find such statements more understandable if they are "translated" into more familiar terms. As students acquire more experience with statements like these, they should be able to dispose with the translations. Here are acceptable translations.

1. For every two numbers, if their sum is non-negative then the sum of their opposites is non-positive. [Be sure to correct a student if he says 'positive' instead of 'non-negative'.]

2. For every two numbers, if their sum is non-positive then the sum of their opposites is non-negative.[1] [Students should see that 1 and 2 say different things. One way to convince them of the difference is to let them see that some instances of 1 (or 2) are not instances of 2 (or 1).]

3. For every two numbers, their sum is equal to the opposite of the sum of their opposites.

4. For every two numbers, the sum of their opposites is equal to the opposite of their sum. [Each instance of Exercise 4 can be derived from an instance of Exercise 3 (and conversely) if one accepts the principle of opposites and the commutative principle for addition. For example:

   (a) $-(-7) + (-5) = -[(-7) + 5]$

   is an instance of Exercise 4 and can be derived from:

   (b) $(-7) + 5 = -[-(-7) + (-5)]$,

   which is an instance of Exercise 3. For, from (b) and part 1

(continued on T. C. 56B)

B. Check each of the following statements.

1. For every x and y, if $x + y \geq 0$ then $-x + (-y) \leq 0$.

2. For every x and y, if $x + y \leq 0$ then $-x + (-y) \geq 0$.

3. For every x and y, $x + y = -[-x + (-y)]$.

4. For every x and y, $-x + (-y) = -(x + y)$.

5. For every x and y, $|x + y| = |-x| + |(-y)|$.

C. Check each of the following statements.

1. For every x and y, $|x + y| = |x| + |y|$.

2. For every x and y, $|x + y| \leq |x| + |y|$.

3. For every x and y, $|x + y| \geq |x| + |y|$.

4. For every x and y, $|x + y| \geq |x| - |y|$.

5. For every x and y, $|x + y| \leq |x| - |y|$.

MULTIPLICATION

In Unit 1 you learned how to multiply directed numbers. Probably you invented a rule which can be stated somewhat as follows:

> The absolute value of the product of two directed numbers is the product of their absolute values. If both numbers are negative, their product is positive. If one number is positive and the other is negative, then their product is negative.

Note that this rule depends upon your knowledge of multiplying absolute values. Since absolute values are positive numbers,

then you must first know how to multiply two positive numbers before you can multiply two negative numbers, or a negative number and a positive number.

Product of two positive numbers

You use your knowledge of numbers of arithmetic in order to multiply two positive numbers. So, we do not state a rule for this case. We merely give illustrations.

$$+5 \times (+2) = +10$$

$$+3\frac{1}{2} \times (+8) = +28$$

Product of two negative numbers

Illustrations:

$$-5 \times (-2) = +10$$

$$-8 \times (-3\frac{1}{2}) = +28$$

Rule:

For every x and y,
if $x < 0$ and $y < 0$ then $xy = \|x\| \cdot \|y\|$.

Product of a positive number and a negative number

Illustrations:

$$-5 \times (+2) = -10$$

$$+8 \times (-3\frac{1}{2}) = -28$$

Rule:

For every x and y,
if $x > 0$ and $y < 0$ then $xy = -(x \cdot \|y\|)$.

Note that if one of the factors (one of the numbers to be multiplied) is 0, you use the principle of 0 to find the product.

Let us check the statement of these rules by applying them to specific cases.

Example 1. Multiply -3 by -6.

Solution. Since both factors are negative numbers, we use the first boxed statement on page 2-57. We substitute '-3' for 'x' and '-6' for 'y' in the expression

'if $x < 0$ and $y < 0$ then $xy = |x| \cdot |y|$'

to obtain the statement

'if $-3 < 0$ and $-6 < 0$ then $-3 \times (-6) = |-3| \cdot |-6|$'

Since $|-3| = +3$, $|-6| = +6$, and $+3 \times (+6) = +18$, then

$$-3 \times -6 = +18.$$

This result is correct. Note that we could have replaced 'x' by '-6' and 'y' by '-3' to obtain the same product.

Example 2. Find the product of -3 and +6.

Solution. We need to use the second boxed statement on page 2-57. Replace 'x' by '+6' and 'y' by '-3'. Then, the expression

'if $x > 0$ and $y < 0$ then $xy = -(x \cdot |y|)$'

becomes, after substitution, the true statement

'if $+6 > 0$ and $-3 < 0$ then $+6 \times (-3) = -[(+6) \cdot |-3|]$'

Since $|-3| = +3$, $+6 \times (+3) = +18$, and $-(+18) = -18$, we know that

$$+6 \times (-3) = -18.$$

EXERCISES

A. Use the boxed statements on page 2-57 to find the products.

1. $-5 \times (-7)$
2. $+2 \times (-9)$
3. -2×5
4. $-8 \times (-8)$
5. $6 \times (-\frac{1}{6})$
6. $-\frac{1}{7} \times (+7)$

B. Check the following statement.

For every x, $-x = (-1)x$.

Do you see that this statement provides you with a rule for finding the opposite of a number? Use the statement to find the opposite of the numbers listed in each of the following exercises.

Students should give derivations for Exercises 5, 6, and 7. They should use the generalizations established in Exercises 1-4, and the fact that, for every x and y, x - y and y - x are opposites. Have them derive this last generalization (see T. C. 53A). Exercise 5 can be handled as follows.

> Since a - b is the opposite of b - a, every instance of Exercise 5 is an instance of:
>
> (1) For every x and y, $xy = -(-x)y$.
>
> Statement (1) was derived in Exercise 1. So, since every instance of Exercise (5) is true, then Exercise 5 is true.

The four generalizations in Part D are false, and students should be able to say this without hesitation. Ask them to restate the generalizations without using such words as 'product', 'positive number', and 'negative number'.

1. For every x and y, $xy > 0$.
2. For every x and y, $(-x)y < 0$.
3. For every x and y, $x(-y) < 0$.
4. For every x and y, $(-x)(-y) > 0$.

When stated as above, the generalizations are "more obviously" false than when stated in the form given in the text.

so, $(-x)(-y) = -[(-x)y]$.

Finally, since $xy = -[(-x)y]$ and $(-x)(-y) = -[(-x)y]$,

$$(-x)(-y) = xy.$$

In justifying this last argument we need no mathematical principles, but only the logical principle according to which, from an equation ['$xy = -[(-x)y]$'] and a second sentence ['$(-x)(-y) = -[(-x)y]$'] we can infer a sentence ['$(-x)(-y) = xy$'] obtained by replacing an occurrence in the second sentence of one side of the equation ['$-[(-x)y]$'] by the other side of the equation. This is an acceptable logical principle because we have decided to use '=' in such a way that an equation such as '$7 \cdot 2 = -[(-7) \cdot 2]$' means that '$7 \cdot 2$' and '$-[(-7) \cdot 2]$' are numerals for the same number.

On the other hand, we can use a mathematical principle derived by using the foregoing logical principle:

For every x, y, and z, if $x = z$ and $y = z$ then $x = y$.

Exercise 2. [This can be derived in the same way as Exercise 1. The following is an alternative derivation using the principle of -1.]

$-xy$	$= (-1)[xy]$	[principle of -1]
	$= [(-1)x]y$	[associativity]
	$= (-x)y$	[principle of -1]
$(-x)y$	$= [(-1)x]y$	[principle of -1]
	$= [x(-1)]y$	[commutativity]
	$= x[(-1)y]$	[associativity]
	$= x(-y)$	[principle of -1]
$-xy = (-x)y$	$= x(-y)$	[principle of equality]

* * *

(continued on T. C. 59C)

In Part C you should inform students of the convention that '-(6)(5)' is an abbreviation for '-[(6)(5)]', that '-(-7)(3)' is an abbreviation for '-[(-7)(3)]', and of the fact that, in general, for every x and y, $-xy = -(xy)$.

Exercises 1, 2, 3, and 4 are easy to establish generally and students should be encouraged to do so. Perhaps you can demonstrate the derivation for Exercise 1 and assign 2, 3, and 4 as homework. You will need to explain the "continued equation" notation. Do this by saying that the statement in 1 is really a short way of making three statements:

For every x and y, $xy = -(-x)y$,

For every x and y, $-(-x)y = (-x)(-y)$,

and: For every x and y, $(-x)(-y) = xy$.

Exercise 1. [To show that $xy = -[(-x)y]$ we can use part 2 of the principle of opposites.]

$$(-x)y + xy = ((-x) + x)y \qquad \text{[distributivity]}$$

$$= 0 \cdot y \qquad \text{[principle of opposites and commutativity]}$$

$$= 0 \qquad \text{[principle of 0 and commutativity]}$$

So, by part 2 of the principle of opposites,

$$xy = -[(-x)y]$$

$$= -(-x)y \qquad \text{[convention].}$$

Similarly,

$$(-x)y + (-x)(-y) = (-x)[y + (-y)]$$

$$= (-x) \cdot 0$$

$$= 0$$

(continued on T. C. 59B)

1. -4 2. -8 3. $-(-5)$

4. $3 + 5$ 5. $-3 + 5$ 6. $5 - 3$

C. Check each of the following statements.

1. For every x and y, $xy = -(-x)y = (-x)(-y)$.

2. For every x and y, $-xy = (-x)y = x(-y)$.

3. For every x, y, and z,
$$xyz = -(-x)yz = (-x)(-y)z = -(-x)(-y)(-z).$$

4. For every x, y, and z,
$$-xyz = (-x)yz = -(-x)(-y)z = (-x)(-y)(-z).$$

5. For every a, b, c, and d,
$$(a - b)(c - d) = -(b - a)(c - d).$$

6. For every a, b, c, and d,
$$(a - b)(c - d) = -(a - b)(d - c).$$

7. For every a, b, c, and d,
$$(a - b)(c - d) = (b - a)(d - c).$$

D. Check each of the following statements.

1. For every x and y,
the product of x and y is a positive number.

2. For every x and y,
the product of $-x$ and y is a negative number.

3. For every x and y,
the product of x and $-y$ is a negative number.

4. For every x and y,
the product of $-x$ and $-y$ is a positive number.

E. Check each of the following statements.

1. For every x and y, $|xy| = |x| \cdot |y|$.

2. For every x and y, $xy \leq |x| \cdot |y|$.

3. For every x and y, $|x| \cdot |-y| = -|x| \cdot |y|$.

4. For every x and y, $|-x| \cdot |y| = -|x| \cdot |y|$.

5. For every x and y, $|-x| \cdot |-y| = |x| \cdot |y|$.

The more able students may appreciate the derivation of the principle of subtraction. [Perhaps they will be able to give it themselves if you show them a derivation of an instance of the principle.]

For each x and y,

$$\begin{aligned}[x + (-y)] + y &= x + [(-y) + y] \\ &= x + [y + (-y)] \\ &= x + 0 \\ &= x.\end{aligned}$$

Since, for each x and y, $x + (-y)$ is a number whose sum with y is x, it follows from the definition of subtraction as the inverse of addition that $x + (-y) = x - y$.

Test the students' understanding of the fact that it is unnecessary to state separate rules for subtracting directed numbers by asking them, for example, to state the rule for subtracting a negative number from a negative number

[For every x and y,

if $x < 0$ and $y < 0$ then $x - y = x + (-y)$]

and to tell which addition rule [page 2-54] covers this case.

SUBTRACTION

When you learned how to subtract directed numbers you noted that a difference of one number from another number could be expressed as a sum of two numbers. For example,

$$+8 - (+2) = +8 + (-2)$$
$$+9 - (-3) = +9 + (+3)$$
$$-5 - (-6) = -5 + (+6)$$
$$-1 - (+1) = -1 + (-1)$$

We can use pronumerals to state this property as follows:

> For every x and y,
> $x - y = x + (-y)$.

Since a difference can be expressed as a sum, every subtraction problem can be solved by considering the corresponding addition problem.

EXERCISES

A. Express each difference as a sum.

1. 3 - 2	2. 5 - 9	3. +6 - (+2)
4. +7 - (+10)	5. 9 - (-2)	6. 3 - (+5)
7. -5 - (-7)	8. -5 - (-1)	9. 8 - 0
10. 0 - (-5)	11. 0 - 0	12. 11 - 11

B. Each of the following expressions stands for a difference (of one number from another) when the pronumerals in the expression are replaced by numerals. Write another expression containing pronumerals which stands for the corresponding sum when the pronumerals are replaced as before.

12. For every a, b, c, and d,

$a - b + (c - d) =$ _____.

13. For every a, b, c, and d,

$a + b - (c + d) =$ _____.

14. For every a, b, c, and d,

$a - 3b - (c - d) =$ _____.

15. For every a, b, c, and d,

$2a + 5b - (-3c + 4d) =$ _____.

16. For every a, b, c, and d,

$7a - (2b - 3c) - (2c - 11d) =$ _____.

17. For every a, b, c, and d,

$(3a + 4b) - (9a - 3c) + (6c - 2d) - (d - 5c) =$ _____.

Each of these two generalizations should be derived.

Exercise 7

$$
\begin{aligned}
x - y - (u - v) &= x - y + [-(u - v)] \\
&= x - y + [(-1)(u - v)] \\
&= x - y + [(-1)u - (-1)v] && \text{[distributivity]} \\
&= x - y + [(-u) - (-v)] \\
&= x - y + [(-u) + v] \\
&= x - y + (-u) + v && \text{[associativity convention]} \\
&= x - y - u + v
\end{aligned}
$$

Then give as supplementary exercises the following. Make true sentences by writing in the blanks expressions which do not contain parentheses.

Sample. $3 + 4 - (6 - 9) =$ _____

Solution. $3 + 4 - (6 - 9) = 3 + 4 - 6 + 9$

1. $7 + 2 + (5 + 3) =$ _____
2. $7 + 2 + (5 - 3) =$ _____
3. $7 + 2 + (-5 + 3) =$ _____
4. $7 + 2 + (-5 - 3) =$ _____
5. $7 + 2 - (5 + 3) =$ _____
6. $7 + 2 - (5 - 3) =$ _____
7. $7 + 2 - (-5 + 3) =$ _____
8. $7 + 2 - (-5 - 3) =$ _____
9. $3 + (4 - 2) - (6 - 7) =$ _____
10. $2 - (3 - 8) - (-7 - 4) - (3 + 9) =$ _____
11. $5 - (6 - 1) - (2 - 5) + (-2 - 6) =$ _____

(continued on T. C. 61C)

[illegible] [illegible]

(continued on [illegible])

[illegible]

[illegible]

[illegible]

The exercises in Part B are very important. Elaborate upon the Sample by giving the following derivation.

$$
\begin{aligned}
& 2x - (-3y) \\
= \; & 2x + [-(-3y)] && \text{[principle of subtraction]} \\
= \; & 2x + 3y && \text{[page 2-59, Part C, Ex. 1]}
\end{aligned}
$$

Then ask students to give a similar demonstration for Exercises 1 and 2. Exercises 3 through 12 should be done at sight. Exercise 13 needs considerable care. Give this demonstration.

$$
\begin{aligned}
k - (m + 5) &= k + [-(m + 5)] && \text{[principle of subtraction]} \\
&= k + [(-1)(m + 5)] && \text{[principle of -1]} \\
&= k + [(-1)m + (-1)(5)] && \text{[distributivity]} \\
&= k + [-m + (-5)] && \text{[principle of -1]} \\
&= k + (-m - 5) && \text{[principle of subtraction]}
\end{aligned}
$$

Exercises 14, 15, and 16 require similar demonstrations.

* * *

The exercises in Part C are also quite important. The ideas in these exercises require more time than is apparent from the number of exercises. Exercise 2 should be derived now if it hasn't been derived before. Restate Exercise 2 as: For every x and y, $-(-x + y) = x - y$, and as: For every x and y, $-x + y$ is the opposite of $x - y$. Add a sixth exercise:

For every x, y, u, and v,

$$x - y - (u + v) = x - y - u - v,$$

and a seventh exercise:

For every x, y, u, and v,

$$x - y - (u - v) = x - y - u + v.$$

(continued on T. C. 61B)

Sample. $2x - (-3y)$

Solution. $2x + 3y$

We can check this second expression by replacing 'x' by, say, '-2' and 'y' by, say, '5' in both expressions and see whether the two expressions then stand for the same number. Upon substitution,

'$2x - (-3y)$'

becomes

'$[2 \times (-2)] - (-3 \times 5)$'

and

'$2x + 3y$'

becomes

'$2 \times (-2) + 3 \times 5$'.

The statement in the box on page 2-60 tells us that

$2 \times (-2) - (-3 \times 5) = 2 \times (-2) + 3 \times 5$.

Do you see that it is correct to say:

For every x and y, $2x - (-3y) = 2x + 3y$.

1. $5a - 3b$	2. $3k - (-2j)$	3. $-7x - (-3y)$
4. $4c - 6d$	5. $5r - 2s$	6. $3a - 7b$
7. $-5z - 3v$	8. $6t - (-7p)$	9. $8x - (-4w)$
10. $10k - (-5)$	11. $0 - (-2x)$	12. $2 - y$
13. $k - (m + 5)$	14. $x - (2y - 3)$	15. $\bigcirc - (\hexagon - \square)$
16. $\square + \bigcirc - (2\hexagon - 7)$		

C. Check each of the following statements.

1. For every x and y, $x - y = -y + x$.
2. For every x and y, $x - y = -(y - x)$.
3. For every x and y, $-x - y = -y - x$.
4. For every x and y, $-x - y = -(x + y)$.
5. For every x, y, u, and v, $x - y + u - v = x + u - y - v$.

Ask students to state a principle of division which is similar to the principle of subtraction. Such a principle of division is:

> For each x and each $y \neq 0$,
> $$\frac{x}{y} = x\left(\frac{1}{y}\right).$$

If they understand the derivation of the principle of subtraction, they should be able to derive the principle of division. Here is one way to do it.

$$\begin{aligned}
\left(\frac{x}{y}\right)y &= x && \text{[division is the inverse of multiplication]}\\
&= x(1) && \text{[principle of 1 for multiplication]}\\
&= x\left[y\left(\frac{1}{y}\right)\right] && \text{[principle of reciprocals]}\\
&= x\left[\left(\frac{1}{y}\right)y\right] && \text{[commutativity]}\\
&= \left[x\left(\frac{1}{y}\right)\right]y && \text{[associativity]}
\end{aligned}$$

Since, for every x and $y \neq 0$, there is just one number whose product with y is x, and since, for every x and $y \neq 0$,

$$\left(\frac{x}{y}\right)y = x = \left[x\left(\frac{1}{y}\right)\right]y,$$

then, for each x and each $y \neq 0$,

$$\frac{x}{y} = x\left(\frac{1}{y}\right).$$

D. Check each of the following statements.

1. For every x and y, $|x - y| \leq |x| + |y|$.
2. For every x and y, $|x - y| \neq |x| - |y|$.
3. For every x and y, $|x + y| - (x + y) \not< 0$.
4. For every x and y, $\big||x| - |y|\big| = |x - y|$.

DIVISION

The rules for dividing directed numbers are easily obtained by using the fact that division and multiplication are inverse operations. Explain.

The quotient of a positive number by a positive number

In this case we use our knowledge of dividing the numbers of arithmetic. For example,

$$\frac{+8}{+2} = +4$$

$$\frac{+3}{+7} = +\frac{3}{7}$$

The quotient of a negative number by a negative number

For every x and y,
if $x < 0$ and $y < 0$ then $\frac{x}{y} = \frac{|x|}{|y|}$.

The quotient of a positive number by a negative number

For every x and y,
if $x > 0$ and $y < 0$ then $\frac{x}{y} = -\frac{x}{|y|}$.

The generalization 'for every x, $0 \cdot x = 0$' is not the principle of 0 for multiplication, but is derivable from it and the commutative principle for multiplication.

* * *

The textbook explanation of the fact that the quotient of 0 by any number (except 0) is 0 is needlessly obscure. All one needs in explaining this fact is the principle of division and the principle of 0 for multiplication. Thus, the quotient of 0 by any number is the product of 0 by the reciprocal of that number, and the principle of 0 tells us that that product is 0.

* * *

Since division is the operation which is inverse to multiplication, for each number x and each number y, x can be divided by y if and only if there is one and only one number z such that $x = y \cdot z$. Hence, division by 0 is not possible since, in case the "dividend" is not 0, there are no candidates for the position of "quotient" and, in case the dividend is 0 there is more than one such candidate.

* * *

Derive the boxed statement in Part B.

$$\frac{x}{-1} = x\left(\frac{1}{-1}\right) \qquad \text{[principle of division]}$$

$$= x(-1) \qquad \text{[fact of arithmetic of directed numbers]}$$

$$= (-1)x \qquad \text{[commutativity]}$$

$$= -x \qquad \text{[principle of -1]}$$

Since this principle will be used in the exercises of Part D on page 2-64, call it for short 'the principle of -1 for division'.

The quotient of a negative number by a positive number

> For every x and y,
> if $x < 0$ and $y > 0$ then $\frac{x}{y} = -\frac{|x|}{y}$.

Note that the quotient of 0 by any number (other than 0) is obtained by using the principle of 0. That is, since we know that

for every x, $0 \cdot x = 0$,

we also know that

> for every x,
> if $x \neq 0$ then $\frac{0}{x} = 0$.

Remember, we cannot divide by 0 because, if we could, the principle of 0 would be violated [Explain].

EXERCISES

A. Check the division rules by finding the quotients.

1. $-8 \div 3$
2. $-6 \div (-2)$
3. $-12 \div (+2)$
4. $\frac{+24}{-8}$
5. $\frac{-7}{-14}$
6. $\frac{-2}{+3}$

B. Check the following statement.

> For every x, $-x = \frac{x}{-1}$.

Do you see that this statement tells you a rule for finding the opposite of a number? Use the statement to find the opposite of the number listed in each of the following exercises.

1. -3
2. 7
3. $-(-2)$
4. $5 + \frac{1}{2}$
5. 0
6. $-[-(-3)]$

Sample 1. $-\frac{-3}{5-2} =$ _____

Solution. $-\frac{-3}{5-2} = \frac{3}{5-2}$

Sample 2. $-\frac{4-7}{6-3} =$ _____

Solution. $-\frac{4-7}{6-3} = \frac{7-4}{6-3}$

1. $-\frac{-8}{5} =$ _____

2. $-\frac{-9}{-7} =$ _____

3. $-\frac{47}{-3} =$ _____

4. $\frac{-8}{-5} =$ _____

5. $-\frac{6-8}{7+5} =$ _____

6. $-\frac{3+5}{2-6} =$ _____

7. $-\frac{-5-6}{-3-2} =$ _____

8. $-\frac{12-3}{8-9} =$ _____

9. $-\frac{7+3-6-2}{2-5+9-3} =$ _____

10. $\frac{6-3-4-8}{7+5+3+1} =$ _____

11. For every a, b, c, and d, if $c - d \neq 0$ then

$$-\frac{a-b}{c-d} = \text{_____}.$$

12. For every a, b, c, and d, $-c - d \neq 0$ then

$$-\frac{a+b}{-c-d} = \text{_____}.$$

13. For every a, b, c, and d, if $c + d \neq 0$ and $a - b - c \neq 0$ then

$$-\frac{(a+b)(b+c)(c+d)}{(c+d)(a-b-c)} = \text{_____}.$$

$$-\frac{x}{-y} = (-1)\left[\frac{x}{-y}\right] \qquad \text{[principle of -1]}$$

$$= \frac{(-1)x}{-y} \qquad \text{[Exercise 1, Part E]}$$

$$= \frac{(-1)x}{(-1)y} \qquad \text{[principle of -1]}$$

$$= \frac{x(-1)}{y(-1)} \qquad \text{[commutativity]}$$

$$= \frac{x}{y} \qquad \text{[Exercise 2, Part E]}$$

$$\frac{-x}{-y} = \frac{(-1)x}{(-1)y} \qquad \text{[principle of -1]}$$

$$= \frac{x(-1)}{y(-1)} \qquad \text{[commutativity]}$$

$$= \frac{x}{y} \qquad \text{[Exercise 2, Part E]}$$

Part D, Exercise 3.

$$-\frac{x(-y)}{uv} = -\frac{-xy}{uv} \qquad \text{[Exercise 2, Part C, page 2-59]}$$

$$= \frac{xy}{uv} \qquad \text{[Exercise 1, Part D]}$$

Use Part C on page 2-59 and Part C on page 2-61 to complete the derivations in Exercises 3 and 4.

* * *

Give these supplementary exercises when Part D has been completed.

Make true sentences out of the following by writing in the blanks expressions containing as few minus signs as possible. [Do not simplify numerators or denominators.]

(continued on T. C. 64F)

Part E, Exercise 3.

$$\left(\frac{x}{y}\right)\left(\frac{u}{v}\right) = \left[x\left(\frac{1}{y}\right)\right]\left[u\left(\frac{1}{v}\right)\right] \quad \text{[principle of division]}$$

$$= \left[xu\right]\left[\left(\frac{1}{y}\right)\left(\frac{1}{v}\right)\right] \quad \text{[associativity and commutativity]}$$

$$= \left(xu\right)\left(\frac{1}{yv}\right) \quad \text{[(*)]}$$

$$= \frac{xu}{yv} \quad \text{[principle of division]}$$

Point out to students that Exercise 2 could have been derived with fewer steps if Exercise 3 was established first.

$$\frac{xz}{yz} = \frac{x}{y} \times \frac{z}{z} \quad \text{[Exercise 3]}$$

$$= \frac{x}{y} \times \left[z\left(\frac{1}{z}\right)\right] \quad \text{[principle of division]}$$

$$= \frac{x}{y} \times 1 \quad \text{[principle of reciprocals]}$$

$$= \frac{x}{y} \quad \text{[principle of 1 for multiplication]}$$

* * *

Part D, Exercise 1.

$$-\frac{-x}{y} = (-1)\frac{(-1)x}{y} \quad \text{[principle of -1]}$$

$$= (-1)\left[(-1)\frac{x}{y}\right] \quad \text{[Exercise 1, Part E]}$$

$$= [(-1)(-1)]\frac{x}{y} \quad \text{[associativity]}$$

$$= (1)\frac{x}{y} \quad \text{[principle of -1]}$$

$$= \frac{x}{y}(1) \quad \text{[commutativity]}$$

$$= \frac{x}{y} \quad \text{[principle of 1]}$$

(continued on T. C. 64E)

Since, for every $x \neq 0$ and every $y \neq 0$, $\left[\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)\right] \times \left[xy\right] = 1$,

then, by the principle of reciprocals, $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$ and xy are reciprocals, that is,

$$\left(\frac{1}{x}\right)\left(\frac{1}{y}\right) = \frac{1}{xy} .$$

(*) can be stated as: For every two non-zero numbers, the product of their reciprocals is equal to the reciprocal of their product.

* * *

Part E, Exercise 2.

$$\frac{xz}{yz} = \left(xz\right)\left(\frac{1}{yz}\right) \qquad \text{[principle of division]}$$

$$= \left(xz\right)\left(\frac{1}{y}\frac{1}{z}\right) \qquad \text{[(*)]}$$

$$= \left[\left(xz\right)\frac{1}{y}\right]\frac{1}{z} \qquad \text{[associativity]}$$

$$= \left\{x\left[z\left(\frac{1}{y}\right)\right]\right\}\frac{1}{z} \qquad \text{[associativity]}$$

$$= \left\{x\left[\left(\frac{1}{y}\right)z\right]\right\}\frac{1}{z} \qquad \text{[commutativity]}$$

$$= \left\{\left[x\left(\frac{1}{y}\right)\right]z\right\}\frac{1}{z} \qquad \text{[associativity]}$$

$$= \left[x\left(\frac{1}{y}\right)\right]\left[z\left(\frac{1}{z}\right)\right] \qquad \text{[associativity]}$$

$$= \left[\frac{x}{y}\right]\left[z\left(\frac{1}{z}\right)\right] \qquad \text{[principle of division]}$$

$$= \left[\frac{x}{y}\right]\left(1\right) \qquad \text{[principle of reciprocals]}$$

$$= \frac{x}{y} \qquad \text{[principle of 1]}$$

(continued on T. C. 64D)

* * *

Exercises 2 and 3 of Part E are easy to derive if we have available the generalization:

(*) For every $x \neq 0$ and $y \neq 0$,

$$\frac{1}{x} \times \frac{1}{y} = \frac{1}{xy} .$$

We derive this generalization here but you should not do so in class until the need for it arises in deriving Exercise 2. Note that (*) is analogous to the generalization:

(♯) For every x and y,

$$(-x) + (-y) = -(x + y)$$

which students should have derived in Part B on page 2-56. Repeat the derivation of (♯) for the students, and encourage them to suggest an analogous derivation for (*). If $x \neq 0$ and $y \neq 0$,

$$\left[\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)\right] \times [xy]$$

$$= \left\{\left[\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)\right]x\right\} y \qquad \text{[associativity]}$$

$$= \left\{\frac{1}{x}\left[\left(\frac{1}{y}\right)x\right]\right\} y \qquad \text{[associativity]}$$

$$= \left\{\frac{1}{x}\left[x\left(\frac{1}{y}\right)\right]\right\} y \qquad \text{[commutativity]}$$

$$= \left\{\left[\left(\frac{1}{x}\right)x\right]\frac{1}{y}\right\} y \qquad \text{[associativity]}$$

$$= \left[\left(\frac{1}{x}\right)x\right]\left[\left(\frac{1}{y}\right)y\right] \qquad \text{[associativity]}$$

$$= \left[x\left(\frac{1}{x}\right)\right]\left[y\left(\frac{1}{y}\right)\right] \qquad \text{[commutativity]}$$

$$= [1][1] \qquad \text{[principle of reciprocals]}$$

$$= 1 \qquad \text{[principle of 1]}$$

(continued on T. C. 64C)

Part D needs considerable work. Each of the four generalizations should be derived. In order to facilitate the derivations, we suggest that you consider Exercises 1, 2, and 3 of Part E. Although students will have no difficulty in <u>verifying</u> these generalizations in Part E, it is instructive to go through the derivations since such derivations should help in dispelling the mystery which surrounds the rules for operating with fractional numbers.

* * *

<u>Part E</u>, <u>Exercise 1</u>.

$$\frac{x}{y}(z) = \left[x\left(\frac{1}{y}\right)\right]z \qquad \text{[principle of division]}$$

$$= x\left[\left(\frac{1}{y}\right)z\right] \qquad \text{[associativity]}$$

$$= x\left[z\left(\frac{1}{y}\right)\right] \qquad \text{[commutativity]}$$

$$= (xz)\frac{1}{y} \qquad \text{[associativity]}$$

$$= \frac{xz}{y} \qquad \text{[principle of division]}$$

$$\frac{xz}{y} = (xz)\frac{1}{y} \qquad \text{[principle of division]}$$

$$= x\left[z\left(\frac{1}{y}\right)\right] \qquad \text{[associativity]}$$

$$= x\left(\frac{z}{y}\right) \qquad \text{[principle of division]}$$

$$x\left(\frac{z}{y}\right) = x\left[z\left(\frac{1}{y}\right)\right] \qquad \text{[principle of division]}$$

$$= (xz)\frac{1}{y} \qquad \text{[associativity]}$$

$$= \frac{1}{y}(xz) \qquad \text{[commutativity]}$$

$$\frac{x}{y}(z) = \frac{xz}{y} = x\left(\frac{z}{y}\right) = \frac{1}{y}(xz) \qquad \text{[principle of equality]}$$

(continued on T. C. 64B)

C. Check each of the following statements.

1. For every x and y, if $y \neq 0$ then $\frac{x}{y} \geq 0$.

2. For every x and y, if $y \neq 0$ then $\frac{-x}{-y} \geq 0$.

3. For every x and y, if $y \neq 0$ then $\frac{-x}{y} \leq 0$.

4. For every x and y, if $y \neq 0$ then $\frac{x}{-y} \leq 0$.

D. Check each of the following statements.

1. For every x and y,
   if $y \neq 0$ then $\frac{x}{y} = -\frac{-x}{y} = -\frac{x}{-y} = \frac{-x}{-y}$.

2. For every x and y,
   if $y \neq 0$ then $-\frac{x}{y} = \frac{-x}{y} = \frac{x}{-y} = -\frac{-x}{-y}$.

3. For every x, y, u, and v, if $u \neq 0$ and $v \neq 0$ then
   $$\frac{xy}{uv} = -\frac{x(-y)}{uv} = \frac{-x(-y)}{uv} = \frac{x(-y)}{u(-v)} = \frac{-x(-y)}{-u(-v)}.$$

4. For every x and y,
   if $x \neq y$ then $\frac{1}{x - y} = -\frac{1}{y - x} = \frac{-1}{y - x}$.

E. Check each of the following statements.

1. For every x, y, and z,
   if $y \neq 0$ then $\frac{x}{y}(z) = \frac{xz}{y} = x(\frac{z}{y}) = \frac{1}{y}(xz)$.

2. For every x, y, and z,
   if $y \neq 0$ and $z \neq 0$ then $\frac{x}{y} = \frac{xz}{yz}$.

3. For every x, y, u, and v,
   if $y \neq 0$ and $v \neq 0$ then $\frac{x}{y} \times \frac{u}{v} = \frac{xu}{yv}$.

(continued on next page)

expression. To set up a rigorous definition of 'well-formed expression' would require that we establish formally the language of mathematics. This kind of activity is one of the things which keeps mathematical logicians busy. Probably the most we can expect of beginners is that they be able to separate algebraic expressions from sentences, separate ill-formed expressions from well-formed ones, and specify which members of the set of directed numbers may not be used in obtaining replacements for the pronumerals in algebraic expressions.

We have not been careful in the students' materials to write 'algebraic expression' instead of 'expression' whenever 'algebraic expression' could be used [See, for example, the instructions in Part A on page 2-67]. Although we are technically correct in omitting 'algebraic', we have found that students objected to the omission. There are good pedagogical grounds for retaining 'algebraic'.

* * *

Your students are likely to consider Exercise 7 an algebraic expression. Call their attention to the next-to-the-last sentence in the paragraph preceding the exercises. Note that when '$\frac{7}{4}$' replaces 'r' in Exercise 7, you do not obtain a numeral. Should we say, then, that:

$$\frac{9 - 3k}{4r - 7}$$

is an algebraic expression? This is the appropriate time to establish the convention with your class that when one regards a given expression as an algebraic expression he automatically eliminates from the set of all replacements for the pronumerals those replacements which would convert the algebraic expression into a symbol which is not a name of a number. In other words, when one says that a given expression is an algebraic expression, he also specifies (implicitly by convention) that the domains of the pronumerals in the expression are just those numbers whose names convert the given expression into numerals. Thus, the domain of 'x' in the algebraic expression '$\frac{1}{x - x}$' is the empty set, and the domain (of real numbers) of 'x' in the algebraic expression '$\sqrt{-x^2}$' is the set consisting of just the number 0.

On the other hand, Exercise 14 is not to be considered an algebraic expression. Logicians would say that Exercise 14 is not a <u>well-formed</u>

(continued on T. C. 65D)

1. For every x, y, and $z \neq 0$, $\frac{x}{z} + \frac{y}{5} =$ _____.

2. For every a, b, $c \neq 0$, and $d \neq 0$, $\frac{a}{c} + \frac{b}{d} =$ _____.

3. For every x, y, z, $u \neq 0$, and $v \neq 0$, $\frac{x}{u} + \frac{y}{v} + \frac{z}{uv} =$ _____.

4. For every $a \neq 0$ and $b \neq 0$, $\frac{1}{a} + \frac{1}{b} =$ _____.

5. For every $x \neq 0$ and $y \neq 0$, $\frac{3}{y} - \frac{2}{x} =$ _____.

6. For every x and $y \neq 0$, $\frac{x}{5} \times \frac{7}{y} =$ _____.

7. For every $x \neq 0$ and $y \neq 0$, $\frac{3}{x} \times \frac{-2}{y} =$ _____.

8. For every $x \neq 0$ and $y \neq 0$, $\frac{7}{x} \div \frac{9}{y} =$ _____.

9. For every x, $y \neq 0$, and $z \neq 0$, $\frac{x}{y} \div \frac{3}{z} =$ _____.

10. For every $a \neq 0$, $b \neq 0$, $c \neq 0$, $\frac{2}{a} + \frac{5}{b} - \frac{7}{c} =$ _____.

* * *

Note our use of the phrase 'algebraic expression'. As we have indicated earlier in this Commentary, we use the word 'expression' to refer to any collection of symbols. We are reserving 'algebraic expression' to be used in referring to what are sometimes called 'terms' in formal descriptions of mathematics. [That is not the way 'term' is used in conventional high school treatments of algebra.] The description of an algebraic expression is to say that it is a collection of symbols (numerals, operation signs, grouping symbols, and pronumerals) which is converted into a numeral by replacing the pronumerals in it by numerals. Expressions other than algebraic expressions are usually sentences whose replacement instances are true or false statements in the arithmetic of directed numbers.

(continued on T. C. 65C)

Note that the justification for:

$$\frac{\frac{x}{y}}{\frac{u}{v}} = \frac{\frac{x}{y} \times \frac{v}{u}}{\frac{u}{v} \times \frac{v}{u}}$$

is the principle of 1 and Exercise 3 of Part E.

* * *

Ask students to justify:

$$\frac{xv}{yv} + \frac{uy}{vy} = \frac{xv + uy}{yv}$$

in Exercise 5 of Part E.

$$\frac{xv}{yv} + \frac{uy}{vv} = \frac{xv}{yv} + \frac{uy}{yv} \qquad \text{[commutativity]}$$

$$= (xv)\left(\frac{1}{yv}\right) + (uy)\left(\frac{1}{yv}\right) \qquad \text{[principle of division]}$$

$$= (xv + uy)\left(\frac{1}{yv}\right) \qquad \text{[distributivity]}$$

$$= \frac{xv + uy}{yv} \qquad \text{[principle of division]}$$

* * *

Supplementary exercises for Part E.

Make the following sentences true by writing a single fraction in each blank.

Sample. For every x, y, and $z \neq 0$,

$$\frac{x}{3} + \frac{y}{z} = \underline{\qquad\qquad}.$$

Solution. $\frac{x}{3} + \frac{y}{z} = \frac{xz}{3z} + \frac{3y}{3z} = \underline{\frac{xz + 3y}{3z}}$.

(continued on T. C. 65B)

4. For every x, y, u, and v, if $y \neq 0$, $u \neq 0$, and $v \neq 0$ then

$$\frac{\frac{x}{y}}{\frac{u}{v}} = \frac{\frac{x}{y} \times \frac{v}{u}}{\frac{u}{v} \times \frac{v}{u}} = \frac{\frac{x}{y} \times \frac{v}{u}}{1} = \frac{x}{y} \times \frac{v}{u}.$$

5. For every x, y, u, and v, if $y \neq 0$ and $v \neq 0$ then

$$\frac{x}{y} + \frac{u}{v} = \frac{xv}{yv} + \frac{uy}{vy} = \frac{xv + uy}{yv}.$$

6. For every x, y, u, and v, if $y \neq 0$ and $v \neq 0$ then

$$\frac{x}{y} - \frac{u}{v} = \frac{xv}{yv} - \frac{uy}{vy} = \frac{xv}{yv} + \frac{-uy}{vy} = \frac{xv - uy}{yv}.$$

2.08 Algebraic expressions--Expressions such as:

$2 \bigcirc + 3 \square$ $\quad 5x + 3y \quad 2k + 7m$

$3abc - 2r \quad 21 + 2w$

$9x - 2x + 7y - 3x \quad 5 \triangle - 7 \square + 2 \triangle - 3 \square$

are called algebraic expressions. Of course, expressions such as these do not stand for numbers until the pronumerals in them have been replaced by numerals. Note that an algebraic expression does not contain a sign of equality or a sign of inequality. When pronumerals in an algebraic expression are replaced by numerals you do not obtain statements; rather, you obtain names for numbers. Numerals are also considered as algebraic expressions.

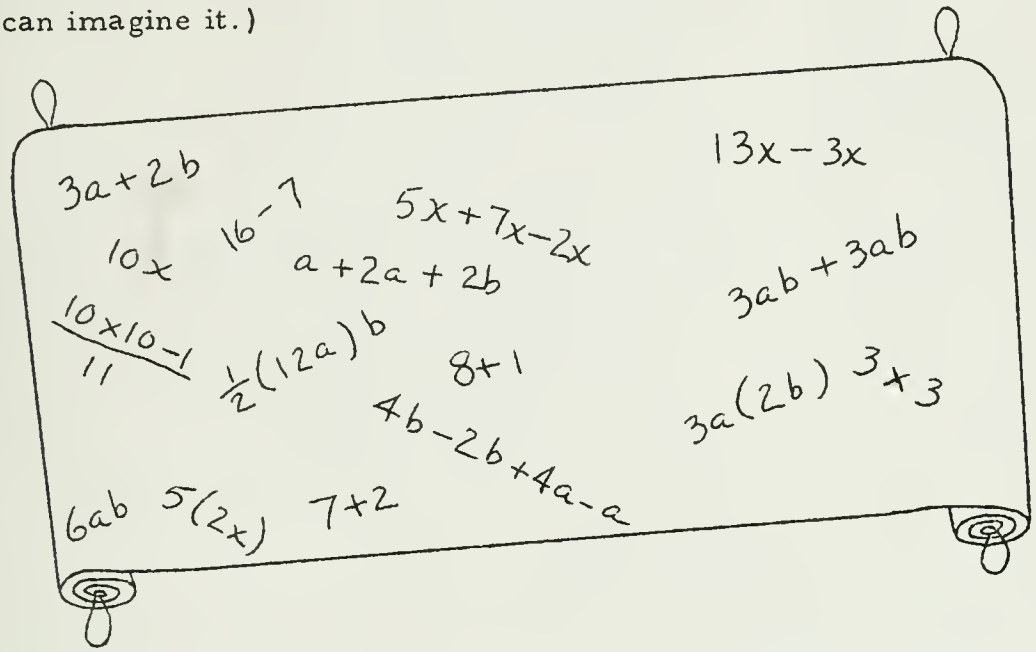
EXERCISES

Tell which of the following are algebraic expressions.

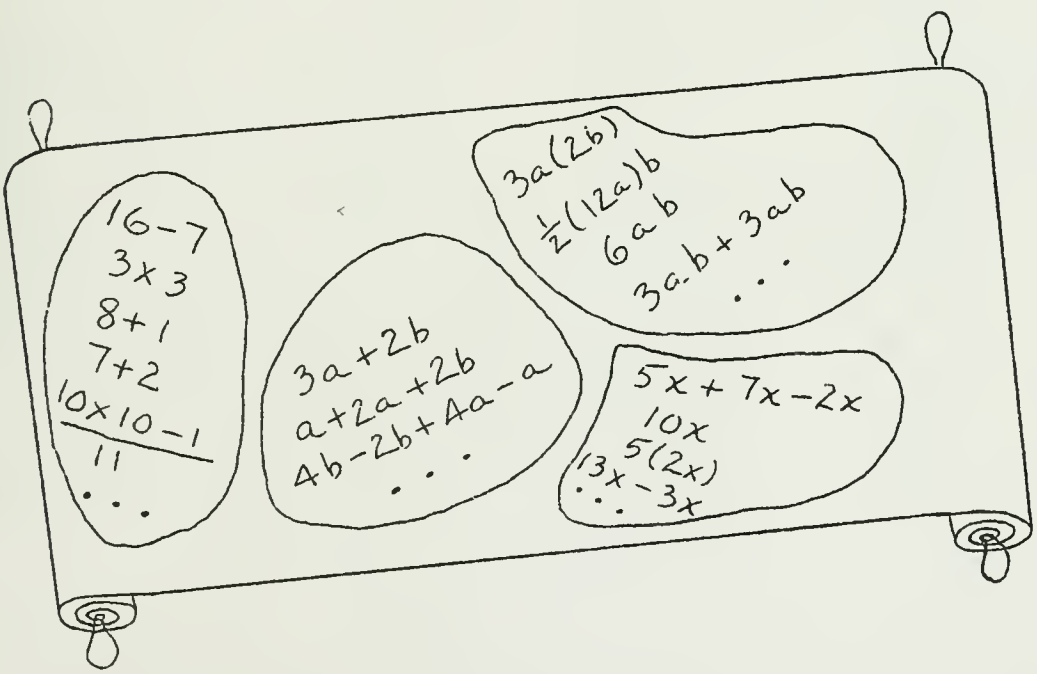
1. $3x - 2y = 7$
2. $5k + 3r - 7$
3. $x - y + x$
4. $7y \geq 4y - 7$
5. $7 \neq 3x$
6. $8t + 7s - 6t$
7. $\frac{9 - 3k}{4r - 7}$
8. $\frac{5j + 2k}{3r} = 5$
9. $x - x$
10. $x - x = 0$
11. $16 + 5$
12. $19 + 3 = 22$
13. y
14. $x \div + [+ (y \div$

EQUIVALENT EXPRESSIONS

Imagine every possible algebraic expression written on a large sheet of paper. (Of course, this really couldn't be accomplished, but you can imagine it.)



Now we can separate these expressions into "families".



example, in 'a + 3b' and 'b + 2a' if one replaces 'a' by '4' and 'b' by '2', one gets two names for 10; there are other replacements for 'a' and 'b', however, which will convert 'a + 3b' and 'b + 2a' into names for <u>different</u> numbers. Clearly, then, the two algebraic expressions are not equivalent.

It is clear that the boxed statement does not tell the student when he has two equivalent algebraic expressions. Nevertheless, he can use the statement to check his guesses about the equivalence of two given algebraic expressions. [Use in class 'equivalent algebraic expressions' rather than 'equivalent expressions'.]

* * *

Students should learn to tell which of two algebraic expressions is simpler-looking merely by looking at each of them. Simplicity of algebraic expressions is a question of esthetics, not of mathematics. As students gain experience, they will also realize that sometimes a complicated-looking algebraic expression is more useful than an equivalent simpler-looking one.

* * *

The work on identifying equivalent algebraic expressions should be highly intuitive at this time. We are preparing the student for more formal work later in the unit. In the meantime give the student considerable freedom in the methods he uses to identify equivalent algebraic expressions. The student should learn to depend heavily upon replacements as a check on his intuition. When a student obtains an algebraic expression which is not equivalent to the given one, show him (or have another student show him) that upon replacement he does not obtain names for the same number.

* * *

Avoid mentioning "like terms" or giving any rules for simplification.

* * *

Be sure that the students understand that two algebraic expressions are not necessarily equivalent merely because one set of replacements for the pronumerals give names for the same number. For

(continued on T. C. 67B)

Can you tell how we separated these expressions into families? Let us examine one of the families in more detail. For example, consider the family:

$$3a + 2b$$
$$a + 2a + 2b$$
$$4b - 2b + 4a - a$$
$$\cdots$$

Suppose 'a' is replaced by a numeral, say, '5' and 'b' is replaced by a numeral, say, '-6'. Then, the expression '3a + 2b' becomes a name for the number 3. The expression 'a + 2a + 2b' becomes a name for the number 3. The expression '4b - 2b + 4a - a' also becomes a name for the number 3. Suppose 'a' is replaced by '7' and 'b' is replaced by '$4\frac{1}{2}$'. Then the expression '3a + 2b' becomes a name for the number 30. Do you think that each of the expressions 'a + 2a + 2b' and '4b - 2b + 4a - a' also becomes a name for the number 30? Check your answer by making the substitutions for 'a' and 'b' in those expressions. The expressions '3a + 2b', 'a + 2a + 2b', and '4b - 2b + 4a - a' are called equivalent algebraic expressions or, merely, equivalent expressions.

> When the pronumerals in equivalent expressions are replaced by numerals, the expressions become names for the same number.

EXERCISES

A. In some of the following exercises two of the three given expressions are equivalent. Tell which two are equivalent and decide which of these two is simpler-looking.

1. $17\frac{1}{2} + 2\frac{1}{2}$
   $10 + 2$
   5×4

2. $5x + 7x$
   $35x$
   $12x$

3. $8y - 2y$
   $6y$
   $10y$

4. $a + 3b$
   $2a - a + b + a$
   $b + 2a$

5. $0 \div x$
   0
   $x \div 0$

6. $c + 7$
   $8c + 7 - 7c$
   $9c - 4 - c$

(continued on next page)

3b + 4a

4a

Exercises 16, 18, and 20 require special mention. Students at Pekin were quick to point out that according to our description of equivalent algebraic expressions, it was not possible to find pairs of equivalent algebraic expressions in these exercises. This observation is correct. However, if you establish the convention about considering only the permissible replacements for the pronumerals in algebraic expressions, then you can find pairs of algebraic expressions in these exercises which are equivalent with respect to the restricted domains. For example, one would write for Exercise 16:

$$\text{For every } a \text{ and } c \neq 0, \quad \frac{6a}{14c} = \frac{3a}{7c},$$

or, in a more abbreviated fashion:

$$\frac{6a}{14c} = \frac{3a}{7c}, \quad [c \neq 0].$$

* * *

The first sentence in the instructions of Part B tells exactly what we mean by the word 'equation'.

* * *

In Part C you should expect the student to be able to state that he can prove that two algebraic expressions are not equivalent by making replacements which give a false statement. The second question in Part C provides a transition to the material immediately following the exercises. We expect the student to say in answer to the second question in Part C that he can't really prove that two algebraic expressions are equivalent by making replacements. What is required in showing equivalence is a transformation of one of the given expressions into the other by using the principles of arithmetic.

7. $4\square - \bigcirc$
$3\bigcirc$
$2\bigcirc + \bigcirc$

8. $x + 7x$
$6x$
$8x$

9. $2a - 3b + 4a$
$-ab + 4a$
$6a - 3b$

10. $3x(2y)$
$6xy$
$5xy$

11. $r(s + t)$
$rs + t$
$tr + sr$

12. $-1(x - y)$
$-x + y$
$-x - y$

13. $-c(2a - 3b)$
$-2ac - 3bc$
$3bc - 2ac$

14. $5r + s$
$s + 5r$
$5s + r$

15. $x(y + 1)$
$xy + x$
$xy + 1$

16. $\frac{6a}{14c}$
$\frac{a}{8c}$
$\frac{3a}{7c}$

17. $-\frac{8}{9}$
$-\frac{8}{-9}$
$-\frac{-8}{-9}$

18. $\frac{k(y + 3)}{k}$
$y + 3$
$ky + 3k$

19. $0(5 - 2x + 3y)$
0
$-2x + 3y$

20. $\frac{x + 1}{y + 1}$
$\frac{x}{y}$
$\frac{3x + 3}{3y + 3}$

21. $a(ab)(3c)$
$(6a)(bc)$
$(2ab)(3ac)$

B. Each of the following exercises contains two expressions separated by ' = '. In some of the exercises the expressions are equivalent. For each exercise tell whether the two expressions are equivalent or not.

1. $3A + 12A = 15A$
2. $3x \times 8y = 24xy$
3. $5b + b = 6b$
4. $10u - 3u = 13u$
5. $11x + 2y = 22xy$
6. $9a - 2b + 3a = 12a - 2b$
7. $3c - 3a = c - a$
8. $12r - r = 11r$
9. $6\Delta - 3\Delta = 2\Delta$
10. $8x \times 2x = 16xx$
11. $7m + 1 = 8m$
12. $4z + 3 = 3z + 3 + z$

C. Suppose you are given two expressions which are not equivalent. How could you <u>prove</u> that they are not equivalent? Suppose you are given two expressions which are equivalent. How could you <u>prove</u> that they are equivalent?

COMBINING TERMS

In the algebraic expression

'$3x + 2y + \frac{1}{2}xz + 3abc + 9$'

the expressions '$3x$', '$2y$', '$\frac{1}{2}xz$', '$3abc$', and '9' are called

Some students may try to explain the equivalence of '3x + 10x' and '13x' by using the erroneous "apple-tree algebra" device: 3 apples + 10 apples = 13 apples. Of course, this analogy is a severe impropriety. An expression such as '3x' is not an abbreviation for '3 things'. It is an expression which becomes a name of a number after the pronumeral has been replaced by a numeral. "Apple-tree algebra" fails in cases like '5x - 7x' and '$\sqrt{3}x + \sqrt{5}x$'.

The only way to justify the equivalence of '3x + 10x' and '13x' is to use the principles of arithmetic. Of course, as students write equivalent expressions in exercises, they are not expected to give the justifications we have given. This situation parallels the one in elementary school where the student ceases to justify each step when he uses the algorism for long-multiplication. As in elementary school, he should be asked once in a while to justify his work.

* * *

You will not find it necessary to introduce such colloquialisms as 'combine like terms' or 'add like terms'. Students will very quickly discover their own shortcuts in simplifying the exercises on page 2-71.

* * *

The justifications for each of the generalizations in line 13 from the bottom and line 10 from the bottom is that all of the instances of these generalizations are instances of the distributive principle.

terms of the algebraic expression or, briefly, terms. In simplifying algebraic expressions we frequently combine terms. For example, an expression which is simpler than but equivalent to the expression '3x + 10x' is '13x'. The two terms '3x' and '10x' are combined into the single term '13x'. Clearly, '13x' is simpler in appearance than '3x + 10x'. But, how do we know that '13x' and '3x + 10x' are equivalent? That is, how do we know that the following statement is true?

For every x, 3x + 10x = 13x.

We might substitute a numeral for 'x' in '3x + 10x = 13x' and find that the resulting statement is true. However, can we be sure that every substitution would result in a true statement? To answer this question we use the principles of arithmetic, principles which we have taken for granted. Let us see how we use the principles of arithmetic to convince ourselves that '3x + 10x = 13x' will result in a true statement no matter what numeral replaces 'x'.

By the distributive principle we know that

for every a, b, and c, a(b + c) = ab + ac.

If we replace 'b' by '3' and 'c' by '10', then we know that 'a(3 + 10)' and 'a3 + a10' are equivalent expressions. That is,

for every a, a(3 + 10) = a3 + a10.

We can say what amounts to the same thing if we replace 'a' by 'x':

For every x, x(3 + 10) = x3 + x10.

Now, if we apply the commutative principle for multiplication, we can say that

for every x,

$$x(3 + 10) = (3 + 10)x,$$

$$x3 = 3x,$$

and $$x10 = 10x.$$

Therefore, we can say that

for every x, (3 + 10)x = 3x + 10x.

Since 3 + 10 = 13, we know that:

For every x, $13x = 3x + 10x$.

Thus, '13x' and '3x + 10x' are equivalent expressions and, of course, '13x' is simpler in appearance than '3x + 10x'.

Now, in combining terms of an expression to get an equivalent expression, it is not necessary to go through all of the steps we did in order to justify your work. You should realize, however, that the principles of arithmetic underlie the steps in your simplifications.

EXERCISES

A. What are the terms in each of the following algebraic expressions?

1. $3x + 17y$
2. $5k + 4z + 3ap$
3. $7t + 5s + (-3r)$
4. $7t + 5s + 3r$
5. $5x + 3$
6. $17 + 8$
7. $A + \frac{1}{2}b$
8. $A + 0$

B. Write an equivalent but simpler expression for each of the following algebraic expressions. Study the samples carefully.

Sample 1. $3x + x$

Solution. We know that for every x, $x = 1x$. Therefore, '3x + x' and '3x + 1x' are equivalent expressions. Since '3x + 1x' is equivalent to '4x' we can say that '4x' is equivalent to '3x + x'.

Sample 2. $7y - y$

Solution. For every y,

$$\begin{aligned} 7y - y &= 7y + (-y) \\ &= 7y + (-1y) \\ &= 6y. \end{aligned}$$

Sample 3. $3a + 5b - 2a - 6b$

Solution. For every a and b,

$$\begin{aligned} &3a + 5b - 2a - 6b \\ &= 3a + 5b + (-2a) + (-6b) \\ &= [3a + (-2a)] + [5b + (-6b)] \\ &= \quad 1a \quad + \quad (-1b) \\ &= a - b. \end{aligned}$$

It is well to discuss Exercise 13 with the class. Be sure that all understand that it can be rewritten as:

$$(-p) + (-3p) + (-2p) .$$

* * *

If your experience this year is anything like the experiences of our teachers in previous years, you will need more exercises like those in Parts B and C. The achievement of a high degree of skill in simplifying algebraic expressions is worth the extra time you will need to put on this. [Two sets of homework paper are provided for Part B, and three sets are provided for Part C.]

* * *

In Part C on page 2-72, you may encounter the usual error made by students when they claim, for example, that '8xx - 6x' and '2x' are equivalent expressions. The best way to handle this, of course, is to resort to replacement of pronumerals by numerals. If a student should suggest 'x(8x - 6)' as equivalent to '8xx - 6x', his suggestion should be accepted.

Sample 4. $5x - y + 7 + 4y$

Solution. For every x and y,

$$5x - y + 7 + 4y = 5x + (-1y) + 7 + 4y$$
$$= 5x + 7 + [(-1y) + 4y]$$
$$= 5x + 7 + 3y.$$

Sample 5. $8x + y + 7$

Solution. There is no equivalent expression which is simpler than '$8x + y + 7$'. Of course, there are many expressions which are equivalent to the given expression [for example, '$7x + x + y + 7$' and '$4x + \frac{1}{2}y + 5 + 4x + 2 + \frac{1}{2}y$'], but none are simpler-looking.

1. $2a + 4a$
2. $7c + 8c$
3. $12p + 17p$
4. $11n + (-7n)$
5. $3x + (-2x)$
6. $.6z + -7z$
7. $(-2)u + 3u$
8. $(-3t) + (+4t)$
9. $9d + d + 6$
10. $-2m + 3m + (-7m)$
11. $8e - 7e + 6e$
12. $12w - w + 3w$
13. $-p - 3p - 2p$
14. $-7f - 3f + 6f$
15. $2g + 0.4g + 0.6g$
16. $\frac{1}{3}b + \frac{1}{5}b + \frac{3}{5}b$
17. $\frac{1}{2}m + \frac{1}{4}m - \frac{1}{8}m$
18. $3a + 6b + 7a$
19. $7x - 2y - 5y$
20. $4m + n + 8m$
21. $9x - x + 3 + 7$
22. $8\Delta - 3\square + 7\square - 2\Delta$
23. $4\bigcirc - \bigcirc - 8\bigcirc$
24. $k + 3k - 5 + 8m$
25. $6a + 11b - 3c$
26. $5L + 2M - 6L$
27. $7x - 3y - 10y + 2x$
28. $a - b - a - 1$
29. $-3b + 7c - 8d - 11b$
30. $5m - 2 - 6m - 7$
31. $-p - 2p - 3p - 4p - 5p$
32. $10r - 7s + 11 - 17r - 19s$
33. $12w - 3z + 6x - 4z - 3w$
34. $\frac{1}{2}a - \frac{3}{7}b + \frac{2}{3}a - \frac{4}{5}b$
35. $\frac{5}{7}c - \frac{1}{2} - \frac{5}{7}c + 1$
36. $1.7d + 7.3e + 6.8d - 10.2e$
37. $-p - 0.5r - 0.7p + 1.5r$
38. $w - 6w + 5 - 2z$
39. $-u + 2u - v + 3$
40. $5k - 3j + 7 + 4a$
41. $5n - 3x - 5n + 3x$
42. $3c - 4d - 4c + 3d$
43. $6g - 3 + 7g - 5h + 12$
44. $1 - 2 + 2a - a$
45. $4 - 6y - 3 - 4 + 6y$
46. $3.6p - 7.2r - 4.2p + r$
47. $1.5 + 10v - w + 2.7w$
48. $4 \times 3 - 3 \times 3 + 2 \times 3$
49. $5 \times 4 - 3 \times 4 + 4$
50. $8 \times 9.6172 + 5 \times 9.6172 + 7 \times 9.6172 - 10 \times 9.6172$

C. Simplify.

Sample 1. $3xx - 2x + 4 + 5xx - 4x + 9$

Solution. For every x,

$$\begin{aligned} & 3xx - 2x + 4 + 5xx - 4x + 9 \\ = & [3xx + 5xx] + [(-2)x + (-4)x] + [4 + 9] \\ = & 8xx + (-6)x + 13 \\ = & 8xx - 6x + 13. \end{aligned}$$

Sample 2. $5abc - 6ab + 7a - 3a + 9ab - 12abc$

Solution. For every a, b, and c,

$$\begin{aligned} & 5abc - 6ab + 7a - 3a + 9ab - 12abc \\ = & [5abc + (-12)abc] + [-6)ab + 9ab] + [7a + (-3)a] \\ = & -7abc + 3ab + 4a. \end{aligned}$$

1. $4xy - 3xy + 7xy$
2. $5abc - 6abc + 7abc$
3. $2xy - 3x + 5xy$
4. $8mp - 5p + 4m$
5. $7xx - 3x + 8xx + 9 - 6x + 12$
6. $5y - 6yy + 18 - 3y + 9yy - 2$
7. $5ab - \frac{1}{2}a - 6b + 9a - 3b + 8ab$
8. $\frac{1}{3} + 7x - 8y + 7xy - \frac{2}{5} + 9x - 6y$
9. $8cd - 6cd + 7c - 5d + 8cd - 6c + 19$
10. $6rst - 7st + 8srt - 9tr + 12srt - 6t + 12st$
11. $7aaa - 3aa + 9a - 8 + 2aaa - 7aa + 5a - 7$
12. $8\frac{1}{2} + 6b - 3bb + 12\frac{1}{2}b - 10\frac{1}{4} - 4\frac{1}{2}bb$
13. $7xy - 5x + 12y - 6xy + 5xx - 9x + 17y$
14. $5.2xyz - 3.5yzx + 18.3yy + 4.8xx - 7.7zz$
15. $8.6 - 3.0aa + 7.9a - 12.6aa + 15.3a - 7.9$
16. $5\Delta\Delta - 2\Delta + 8 - 7\Delta\Delta + 5\Delta - 6$
17. $1 + x + xx + xxx + xxxx$

D. Use the simplest expressions you can find to complete the following.

Sample. For every x, if a side of an equilateral triangle is $2x - 3$ inches long, the perimeter is _____ inches.

Solution. The perimeter of a triangle is found by adding the lengths of the three sides, and in an equilateral triangle the three sides have the same length.

In handling the exercises in Part D and others of this type throughout the unit, you may want to spend a little time in working with the concept of the domain of a pronumeral. For example, in the Sample for Part D since (according to an agreement stated on page 2-33) the domain of 'x' is the set of all directed numbers, in order for the completed statement to make sense, it should go something like this:

> For every x, if $x \geq \frac{3}{2}$, and if a side of an equilateral triangle is 2x - 3 inches long, the perimeter is _____ inches.

You can handle this matter in an informal way by asking a question such as:

> What is the smallest number whose name can replace 'x' and still give a sensible statement?

An alternative way of handling the matter is to restrict the domain of the pronumeral by altering the quantifier. For example:

> For every $x \geq \frac{3}{2}$, if a side of an equilateral triangle is 2x - 3 inches long, the perimeter is _____ inches.

* * *

If your students have difficulty with the exercises of Part D, you may want to turn back to pages 2-23 and 2-24 for an oral review of the exercises there. See, especially, Exercises 2, 6, 7, and 8.

For every x, if a side is 2x - 3 inches long, the perimeter is 2x - 3 + 2x - 3 + 2x - 3 inches. An expression which is equivalent to '2x - 3 + 2x - 3 + 2x - 3' but simpler-looking is '6x - 9'. Therefore, you should write '6x - 9' in the blank space.

1. For every a and c, if a side of a square is 2a - 4c feet long, the perimeter is _____ feet.

2. For every r and s, if a side of an equilateral triangle is 7r - s inches long, the perimeter is _____ inches.

3. For every x and y, if a rectangle is 2x - y inches long and -x + 2y inches wide, the perimeter is _____ inches.

4. For every a and c, if each of two sides of an isosceles triangle is 2a - c inches long, and the base is a - c inches long, the perimeter is _____ inches.

5. For every x and y, if the perimeter of an equilateral triangle is 9x - 3y inches, each of its sides is _____ inches long.

6. For every u and v, if the perimeter of a square is 2u - 8v feet, each of its sides is _____ inches long.

FURTHER SIMPLIFICATIONS

You have learned to simplify certain kinds of algebraic expressions by combining terms. In simplifying expressions you obtained equivalent expressions because you used the principles of arithmetic. There are other procedures for simplifying algebraic expressions. We shall now consider some of these procedures.

Example 1. Simplify: $3(2x)$.

Solution. By the associative principle for multiplication we know that

for every a, b, and c, $(ab)c = a(bc)$.

If we replace 'a' by '3' and 'b' by '2' in the expression '$(ab)c = a(bc)$', we have:

For every c, $(3 \times 2)c = 3(2c)$.

Now, we can use any other pronumeral, say 'x', to express the same idea:

For every x, $(3 \times 2)x = 3(2x)$.

Since $3 \times 2 = 6$, we can write:

For every x, $6x = 3(2x)$.

Therefore, '$6x$' is equivalent to '$3(2x)$'. Of course, '$6x$' is also simpler-looking than '$3(2x)$'.

Example 2. Simplify: $3p(5q)$.

Solution. For every p and q,

$$\begin{aligned} 3p(5q) &= [(3p)5]q \\ &= [3(p5)]q \\ &= [3(5p)]q \\ &= [(3 \times 5)p]q \\ &= [15p]q \\ &= 15(pq) \\ &= 15pq. \end{aligned}$$

Can you give a reason for each of the steps in the simplification process? In practice, you should be able to state that '$15pq$' and '$3p(5q)$' are equivalent without going through the intermediate steps.

A. Simplify.

1. $2(7x)$	2. $3(-4a)$	3. $5(-15b)$
4. $6(0t)$	5. $4(9d)$	6. $8(\frac{1}{4}r)$
7. $3(4a)$	8. $7(5x)$	9. $(2b)6$
10. $(3m)7$	11. $2(-8r)$	12. $7(-5s)$
13. $-2(9x)$	14. $-4(-2y)$	15. $-3(-2p)$
16. $-5d \times 12$	17. $(-6z) \cdot 11$	18. $-12d(-5)$
19. $1.2(5a)$	20. $5(1.2a)$	21. $(1.2a)5$
22. $(1.2 \times 5)a$	23. $-1.3(-2e)$	24. $-5.5(3m)$
25. $\frac{1}{5}(10x)$	26. $20y \times \frac{1}{4}$	27. $-21z \times \frac{1}{3}$
28. $\frac{1}{6}(-24r)$	29. $\frac{1}{2}(-34s)$	30. $(-36t)(-\frac{1}{18})$
31. $-7 \times (-12u)$	32. $(-16v)(-4)$	33. $(-12w) \times 7$
34. $-17 \times 4t$	35. $-2.3(1.5x)$	36. $1.8(-2.5y)$
37. $5a(3b)$	38. $-3\Delta(4 \square)$	39. $8x \times 2x$
40. $3(-2b)(5c)$	41. $2m(-3)(-2n)$	42. $2t(-2t)$
43. $-3a \cdot 2b$	44. $2a \times (-2b)5c$	45. $a \cdot 3x \cdot 5y$
46. $(-k)(-k)(-k)$	47. $2mn(-3mn)$	48. $3\Delta(2 \square)(-6\Delta)$
49. $1.2 \times 5e \times 6.5$	50. $(8t)(8t)(8t)$	51. $-3.3(4.1xy)$
52. $\frac{1}{2}x \times \frac{1}{3}y$	53. $-\frac{1}{4}y(\frac{2}{3}x)(\frac{6}{7}y)$	54. $(-\frac{3}{5}s)(4t)(-1\frac{1}{3}r)$

B. Complete each of the following.

1. For every x, if one pencil costs 5x cents, then you must pay _____ cents for seven pencils.

2. For every t and r, if one side of a square is 4tr inches long, the perimeter is _____ inches.

(continued on next page)

angle

Here are supplementary exercises for Part B.

1. For every $b > 0$, the area of a circle whose radius is $\frac{b}{2}$ inches long is _____ times the area of a circle whose radius is $3b$ inches long.

2. For every $d > 0$ and $e > 0$, if a rectangle is d inches long and $2e$ inches wide, its perimeter is _____ times the perimeter of a rectangle whose length is $\frac{d}{4}$ inches and whose width is e inches.

3. For every $w > 0$, the circumference of a circle whose radius is ww inches is _____ times the circumference of a circle whose radius is w inches.

4. For every $h > 0$ and $j > 0$, if a parallelogram has a height h feet long and a base $3j$ feet long then its area is _____ times the area of a parallelogram with a height $\frac{h}{6}$ feet long and a base $9j$ feet long.

5. For every $r > 0$ and $w > 0$, if a T.V. set costs $7r$ dollars, has a marked price of w dollars, and is being sold at a discount of 20% of the selling price, the selling price is _____ dollars. The selling price is _____ dollars more than the cost price.

6. For every $r > 0$ and $w > 0$, if a set of 12 chairs costs $3r(r + 2) - w(w + 2)$ dollars and the marked price is $3rr - ww + r + 4w$ dollars, the profit on the set of chairs is _____ dollars. The profit on one chair is _____ dollars.

* * *

In connection with Example 4, let students compare the algebraic expressions: $\frac{1}{2}x$, $\frac{1}{2x}$, $\frac{1}{2} \times \frac{1}{x}$, $\frac{x}{2}$, and: $\frac{2}{x}$.

3. For every x and y, if the dimensions of a rectangle are 7x inches and 3y inches, the perimeter is _____ inches and the area is _____ square inches.

4. For every x and y, if a man can walk a mile in 2xy minutes then he can walk 6xy miles in _____ minutes at the same rate.

5. For every a, b, and c, if there are 20ab sheets of paper in a pile 1 inch thick then there are _____ sheets of the same kind in a pile 15abc inches thick.

6. For every x and z, if oranges cost -5z cents per dozen then 4x dozen oranges cost _____ cents.

7. For every p, r, and t, if 30p dollars are borrowed at an annual interest rate of 5r cents per dollar, the total interest due at the end of 2t years is _____ dollars.

Example 3. Simplify: $35a \div 7$.

Solution. Since division and multiplication are inverse operations, we can say:

For every a,

$$35a \div 7 = 35a \times \frac{1}{7}$$
$$= (35 \times \frac{1}{7})a$$
$$= 5a.$$

Example 4. Simplify: $\frac{14xy}{2x}$.

Solution. For every x and y, if $x \neq 0$, then

$$\frac{14xy}{2x} = 14xy \times \frac{1}{2x}$$
$$= 14xy \times \frac{1}{2} \times \frac{1}{x}$$
$$= (14 \times \frac{1}{2})(x \times \frac{1}{x})y$$
$$= (7)(1)y$$
$$= 7y$$

Do not insist that students go through the intermediate steps as we have done in Example 4 on page 2-76 in simplifying the expressions in Part C. Of course, if they have trouble with certain exercises, you may want to go through the process step-by-step with them for those exercises.

* * *

Students should note the "zero-exceptions" [e.g., in Exercise 13].

* * *

Here again you may need to review the principle of reciprocals and discuss division as the inverse of multiplication.

C. Simplify.

1. $24x \div 6$
2. $12y \div 4$
3. $36a \div 18$
4. $-18b \div 9$
5. $18b \div (-9)$
6. $-18b \div (-9)$
7. $\frac{36u}{9}$
8. $\frac{-12v}{3}$
9. $\frac{-9.9e}{3}$
10. $\frac{-17d}{17}$
11. $\frac{-64a}{-16}$
12. $\frac{-6\square}{6}$
13. $\frac{15xy}{3x}$
14. $\frac{20aab}{-4ab}$
15. $\frac{9xy}{3uv}$
16. $\frac{24xxyz}{6xyyz}$
17. $\frac{5xyz}{5xyz}$
18. $\frac{10\Delta\Delta\square}{.1\Delta\square\square}$
19. $6xy \div (3xyy)$
20. $17a \div (-a)$
21. $9aa \div (-9aa)$

D. Complete the following.

1. For every x, if 10x candy bars cost 50x cents then 1 candy bar costs _____ cents.

2. For every m and n, if the perimeter of a square is 12mn inches, each of its sides is _____ inches long.

3. For every x and y, if it takes a man 3xy minutes to walk 21xxy miles then his average rate in miles per minute is _____.

4. For every x, y, and z, if 36xxyzz apples are to be shared equally among 2xyz children each child should receive _____ apples.

5. For every m and n, if a pile of typing paper containing 70mmn sheets of paper is 3.5mn inches high, a pile of 5mnn sheets of the same typing paper is _____ inches high.

EXPANDING AN EXPRESSION

As you know, an expression is simplified by writing an equivalent expression which is simpler in appearance than the given expression, that is, the simpler expression contains fewer symbols. You are learning how to simplify expressions because simple expressions are sometimes more useful than complicated expressions. However, you may be given an expression and asked to write an expression which is equivalent to but not necessarily simpler than the given expression. Often we use the word expand when we want you to change an expression which contains parentheses into an equivalent one which does not contain parentheses.

Example 1. Expand: $5(3y + 2z)$.

Solution. By the distributive principle we know that for every a, b, and c, $a(b + c) = ab + ac$. If we replace 'a' by '5', 'b' by '3y', and 'c' by '2z', we obtain:

$$5(3y + 2z) = 5(3y) + 5(2z)$$

and we can simplify the expression on the right of ' = ' to obtain:

$$15y + 10z$$

Example 2. Expand: $(12x - 9y) \div 3$.

Solution. For every x and y,

$$\begin{aligned}(12x - 9y) \div 3 &= \tfrac{1}{3}(12x - 9y) \\ &= \tfrac{1}{3}[12x + (-9y)] \\ &= \tfrac{1}{3}(12x) + \tfrac{1}{3}(-9y) \\ &= 4x - 3y.\end{aligned}$$

Example 3. Simplify: $3(2x - 4y) + 5(6x - 7y)$.

Solution. First, we expand '$3(2x - 4y)$' and '$5(6x - 7y)$'. Then we simplify by combining terms:

For every x and y,

$3(2x - 4y) + 5(6x - 7y)$

$= 6x - 12y + 30x - 35y$

$= 36x - 47y.$

Example 4. Simplify: $3(4m - s) - (2m - 4s)$.

Solution. For every m and s,

$3(4m - s) - (2m - 4s)$

$= 3(4m - s) + [-(2m - 4s)]$

Now, we have learned that for every m and s, $-(2m - 4s)$ is the opposite of $2m - 4s$. Therefore, we can use one of our rules for finding the opposite of a number. For example:

For every m and s, $-(2m - 4s) = (-1)(2m - 4s)$.

Therefore,

For every m and s,

$3(4m - s) - (2m - 4s)$

$= 12m - 3s + [-1(2m - 4s)]$

$= 12m - 3s + [-2m + 4s]$

$= 12m - 3s - 2m + 4s$

$= 10m + s$

EXERCISES

A. Expand. (That is, write an equivalent expression which does not contain parentheses or other symbols of grouping.)

1. $3(3x + 2y)$
2. $(a + 4b) \times 5$
3. $4(u - 5v)$
4. $(9m - 3n) \times 7$
5. $\frac{1}{2}(2p + 4r)$
6. $\frac{2}{3}(3s - 12t)$
7. $(16c - 28d) \div 4$
8. $(-14e + 21f) \div 7$
9. $\frac{-25z - 15w}{-5}$
10. $\frac{32h - 8v}{-8}$
11. $6(3x + 2y)$
12. $9(2a + 5b)$
13. $(6u + 10v)5$
14. $(w + 3z)7$
15. $10(5s - 6t)$
16. $8(8c - 9d)$
17. $(3m - 2n)(-4)$
18. $-5(3g - 2h)$
19. $-(10y - 3x)$
20. $-(5x - 1)$
21. $-\Delta(3 \square - 4 \diamond)$
22. $-10(-\Delta - \diamond)$

(continued on next page)

Several teachers reported that students missed the even-numbered problems in Part B. Example 4, page 2-79, is illustrative of this type of exercise. It may be helpful to review Example 4 with the class before they start working Part B. [Two sets of homework paper are provided for Part B.]

* * *

Here are supplementary exercises of a slightly different form which may help.

From each set of expressions pick out all that are equivalent to the first expression.

1. $9(a - b) - 2(a + b)$
   - $7a - b$
   - $9a - (9b + 2a) - 2b$
   - $2(a - b) - 9(a + b)$

2. $3(r - s) - 5(s - r)$
   - $3(r - s) + 5(s + r)$
   - $5(s - r) - 3(r - s)$
   - $5(r - s) + 3(r - s)$

3. $6x + 3y - 9x - 15y$
   - $-3(x + 4y)$
   - $3y + 6(x - y) - 9(x + y)$
   - $3(2x + y) - 3(3x - 5y)$

4. $2(3m - 2n) - 3(6m + 2n)$
   - $-2(6m + 5n)$
   - $\frac{2}{3}(9m - 6n) + \frac{3}{4}(8n - 24m)$
   - $6m - (18m - 2n) + 6n$

5. $3x(x - 2x) - 6 + 9x$
   - $3x(x - 2x - 6) + 9x$
   - $3xx - (6xx - 9x) - 6$
   - $3x(x - 7) - 6$

6. $\frac{1}{5}(25a - 10c) - \frac{1}{5}(5a - 15c)$
   - $\frac{1}{5}(25a - 5c - 10a - 15c)$
   - $(5a - 2c) - (a - 3c)$
   - $4a + c$

7. $-(p + 5q)2 - (p - 2q)5$
   - $-2p - 5p$
   - $-(20q + 7p)$
   - $(10q - 2p) - 5p + 10q$

23. $-11(-3s + 2t)$
24. $-12(-4u - 3v)$
25. $\frac{1}{2}(2c - 4d)$
26. $(9f - 12g) \times \frac{1}{3}$
27. $3x(2a - 3b)$
28. $-5m(2j - 4k)$
29. $-2m(3m - 6n)$
30. $7t(4r - 5t)$
31. $6xy(-3x + 4y)$
32. $-2.5a(1.7a - 3.6b)$
33. $\frac{2}{3}(-9c + 3d)$
34. $\frac{1}{7}(7p - 14r)$
35. $\frac{3}{5}(15x - 25y)$
36. $(5 - 10a)\ \frac{4}{5}$
37. $(18d - 6e) \div 2$
38. $(5m - 12) \div 5$
39. $(3x - 5) \div (-3)$
40. $(4.6x + 6.9) \div 2.3$
41. $(a + 2) \div 2$
42. $(3m - 3) \div 3$
43. $\frac{x - y}{4}$
44. $\frac{2u + 2v}{-4}$
45. $\frac{3ab - 7ac}{a}$
46. $\frac{8xyyz + 12xxyzz}{4xyz}$
47. $\frac{9mmnn - 3mmn}{3mmn}$
48. $\frac{2ap - 7apq}{-ap}$
49. $(y + 5z) \div \frac{1}{2}$
50. $(2a - b) \div \frac{1}{3}$

B. Simplify.

1. $2(x + 2y) + 3(x - 2y)$
2. $5(x + 2y) - 3(x - y)$
3. $4(a - 3b) + 2(a + 2b)$
4. $4(2a + 3b) - 4(3a + 2b)$
5. $7(u - v) + 6(v - u)$
6. $7(3m - 4n) - (17m + 24n)$
7. $(3c - 5d) \times 5 + (3d - 2c) \times 6$
8. $8p - 3r - 2(4p - 5r)$
9. $-3(\Delta + 2 \diamondsuit) + 2(4 \diamondsuit - 3\Delta)$
10. $(3 \square - 4\Delta) \times 5 - 6(-4\Delta + \square)$
11. $9(7 - 2t) + 4(3 - t)$
12. $10(c + d) - 20(c + d)$
13. $6(3s - 5) + 4(3 - 5)$
14. $9(-u + 2g) - 8(-u + h)$
15. $3x(x - 5) + 7x(2x - 8)$
16. $2y(y - 7) - 5y(4 - 2y)$
17. $5ab(6a - 2b) + 2ab(5b - 8a)$
18. $3mn(2m - 5n + 4) - 2mn(6 - 3m + 4n)$
19. $\frac{1}{4}(8e - 4g) + \frac{1}{2}(2g - 6e)$

(continued on next page)

2. For every $a > 0$, $m > 0$, and $b > 0$, if the base of a right triangle is $\frac{a}{m}$ and the altitude is $\frac{b}{mm}$, then its area is _____ .

3. For every $d > 0$ and $t > 0$, a car travels $100d$ miles at a rate of $\frac{10}{d}$ miles per hour in _____ hours. The car travels $250d$ miles at a rate which is $\frac{1}{4}$ of its former rate in _____ hours.

4. For every r, q, m, and n, if the amount of money put in the bank is $5rq$ dollars, and the rate of interest is $\frac{.5m}{20n}$, the (simple) interest earned for 1 year is _____ dollars. The (simple) interest for n years is _____ dollars, (simple) interest for $\frac{1}{n}$ years is _____ dollars. If the total (simple) interest earned is $\frac{.5rqm}{4}$ dollars, the principal has been saved for _____ years.

5. For every △ , ○ , and ⬡ , if △ dozen stamps cost 2○ cents, then the cost in cents of 2 dozen stamps is _____ . The cost of $\frac{1}{4}$ dozen stamps is _____ cents. The cost of ⬡ stamps is _____ cents.

6. For every □ , ○ and △ , if □ feet of ribbon cost 3○ cents, then the cost of △ inches of ribbon is _____ cents.

Here are more supplementary exercises for Part B.

Simplify the following expressions:

1. $2(x - 3y) + 5(3y - x)$
2. $14(w - u) + 7(u - w)$
3. $(3a - 5b) - (5b - 3a)$
4. $2m - 5n - (4n - 10m)$
5. $\frac{1}{3}(r - s) - \frac{1}{2}(s - r)$
6. $3(x - 5) - 5(5 - x)$
7. $\frac{2}{3}(3xy - z) + \frac{2}{5}(z - 3xy)$
8. $6(m - n) - 3(2m - n) + \frac{2}{3}(n - m) + 4(m - n)$
9. $\frac{1}{3}mnn(3mmnn - 2mmn) - \frac{1}{9}mnn(4mmn - 6mmnn)$
10. $2(sst - tt) + (ss - stt) - 3(t - s)$
11. $(3xxx - xy) + (5xxy - xy) - 6(yy - xyy) - 2(yy - x)$
12. $\frac{2(r - s)}{s - r}$
13. $\frac{-3(m - n)}{2(n - m)}$
14. $\frac{5y - 3x}{4} - \frac{3x - 5y}{2}$
15. $\frac{2(u - v)}{7(v - u)} + \frac{-5(v - u)}{u - v}$

* * *

Here are supplementary exercises similar to those of Part C [pages 2-81-82].

Complete the following statements, then replace pronumerals by numerals to determine whether the expression you wrote in the blank gives a true instance.

1. For every $k > 0$, if the height of a triangle is $2k$ times the height of another triangle, and the base of the first triangle is $\frac{1}{4k}$ times the base of the second, then the area of the first triangle is _____ times the area of the second triangle.

(continued on T. C. 81B)

20. $\frac{1}{2}(2a + 4b) - \frac{1}{3}(6a + 9b)$

21. $\frac{1}{3}(9w + 12) + \frac{2}{5}(-15 - 5w)$

22. $\frac{1}{3}(12p - 9c) - \frac{1}{2}(4p - 6c)$

23. $\frac{1}{5}x(\frac{2}{3}x - \frac{1}{2}y) + \frac{1}{3}y(\frac{2}{5}y - \frac{2}{3}x)$

24. $\frac{a}{6}(12a - 18b) - \frac{b}{3}(24b - 9a)$

25. $\frac{-4c - 12d}{4} + \frac{9c - 3d}{3}$

26. $\frac{-4c + 16d}{2} - \frac{-4c + 16d}{4}$

27. $\frac{2p - 4a}{4} + \frac{4p - 8a}{8}$

28. $\frac{14u - 7}{7} - \frac{9a - 9}{9}$

29. $(4.2s - 6.4t) \div 2 + (1.8t - 3.6s) \div 3$

30. $(.27x + 9y) \div 9 - (18x + 36y) \div 2$

C. Complete the following.

1. For every x, if one side of a square is x + 4 inches long, the perimeter is _____ inches.

2. For every a and b, if one side of a square is 8ab - 3a + 7b inches long, the perimeter is _____ inches.

3. For every x and m, if a rectangle is 5x - 2m inches long and 3x + 2m inches wide, the perimeter is _____ inches.

4. For every a, b, and c, if a rectangle is 5a - 3b + 6c feet long and 6a - 7b - 3c inches wide, the perimeter is _____ inches.

5. For every r, s, and t, if a rectangle is 3r - 4s inches long and 5t inches wide, the area is _____ square inches.

6. For every t and b, if a side of an equilateral triangle is 5t - 3b inches long, the perimeter is _____ inches.

7. For every x and y, if the base of an isosceles triangle is x - y inches long and if each of the other two sides is 2y - x inches long, ther perimeter is _____ inches.

(continued on next page)

The exercises in Part A of the Review Exercises should be worked on an intuitive basis. In a sense, this Part serves as a transition to the next unit on equations. We think most students will be able to handle these exercises with little difficulty. In fact, after checking the first two exercises with the class, you may want to give the rest of the exercises as an informal test. If you do, we should appreciate your sending us the test scores.

8. For every y and x, if the cost of a book is 2y - 3 dollars, then the total cost of 7x such books is _____ dollars.

9. For every i and j, if the perimeter of a square is 24i - 4j inches, each of its sides is _____ inches long.

10. For every v, if the perimeter of an equilateral triangle is 9v - 4v inches, each of its sides is _____ inches long.

11. For every a, b, and c, if the area of a rectangle is 18ab - 30bc square inches and is 3b inches wide then it is _____ inches long.

12. For every p and r, if a train travels 48ppr - 16prr miles in 4pr hours, its average speed is _____ miles per hour.

REVIEW EXERCISES

In this set of exercises you will find problems which help you review what you have learned. Also, you will find exercises reviewing your general knowledge of mathematics. Some exercises will teach you something you did not previously know. For each exercise you should ask yourself, "Should I have learned this in the unit, did I already know it, or am I learning something new?"

A. In each exercise write a numeral for the same number in place of each pronumeral so that the completed statement is true.

1. $4 \square + 1 = 9$
2. $8 \diamond - 2 = 38$
3. $2 \bigcirc + \bigcirc = 12$
4. $9\Delta - 2\Delta = 28$
5. $3x - 17 = 1$
6. $2y + 12 = 18$
7. $5 - 3A = -7$
8. $20 + 3B = 50$
9. $2m + 5m = 35$
10. $p - 4p = 17$
11. $\frac{1}{2}y = 12$
12. $50\%x = 100\%$
13. $\frac{1}{2}x + 5 = 75$
14. $7 + \frac{1}{3}y = 15$
15. $\frac{x - 4}{3} = 12$
16. $\frac{5 - y}{9} = 2$
17. $\frac{2A + 1}{4} = 8$
18. $\frac{12 - 7k}{2} = 5\frac{1}{2}$

B. For each of the following expressions write three expressions which are equivalent to it.

1. $3x + 2y + 9$
2. $x + 5x - 7$
3. $5 + 9 - 6$
4. $18 - 2x + 25$
5. $6 \square - 3\Delta$
6. $4\Delta\Delta + 2\Delta$
7. $4x(2y - 3z)$
8. $2a(3a + 6aa)$
9. $-3[-2 + (-12)]$
10. $\frac{1}{2}x - \frac{2}{3}y$
11. $27xy$
12. $(33xy) \div (3x)$

C. Each of the following expressions contains one or more pronumerals. For each expression write another expression which contains the same pronumerals but is not equivalent to the given expression. Prove that the given expression and the one you have written are not equivalent.

1. $2x + 1$
2. $3yz$
3. $6\Delta - 2\Delta$
4. $7 \square \times 3 \diamond$
5. $3xx - 2x$
6. $7 + 9.5x$
7. $10x - 10x$
8. $\frac{1}{2}a + \frac{1}{2}b$
9. $\frac{2}{3}(\frac{3}{5}xy)$
10. $(0.14xxyyy) \div (2xyy)$

D. Write the simplest names you can for the numbers named by the following expressions.

1. $-5 + 7 + (-3)$
2. $6 + (-6) + (-2) - 5$
3. $15 - 8 + 9 - 17$
4. $5 + 6 - 3 - 0 + (-20)$
5. $12 - 7 + 9 - 6 - 3 + 4 - 17 + 8 - 3 + 2$
6. $-6 - 8 + 9 - 2 - 5 - 3 + 7 - 6 + 4 - 12$
7. $-6\frac{1}{2} - 12\frac{1}{4} + 5\frac{1}{2}$
8. $16\% - 38\% + .42$
9. $6(5 - 3 + 7) - 2(4 + 3 - 12)$
10. $-7[+6 - (-3)] - 2[-3 + (-7)]$

(continued on next page)

11. $-6(5)(-2) + 4(-3)(-2)(-1)$

12. $(-1)(-1)$

13. $(-1)(-1)(-1)$

14. $(-1)(-1)(-1)(-1)$

15. $(-1)(-1)(-1)(-1)(-1)$

16. $-3\{(-10) \times (-7) + 6(-3) - (-4)[-2(-5) + (-3)(-7)]\}$

17. $\dfrac{8(-2)(-5) + 7(-3)(5)}{4(-6)(+5) - 3(8)(-2)}$

18. $\dfrac{7(-8)(-2) - 6(-2)(+4)}{5(3)(-6) + 4(-2)(+5)}$

19. $5(-3)(-3) - 6(-3) - 63$

20. $5(4.2)(4.2) - 6(4.2) - 63$

E. Write an equivalent expression without parentheses and then simplify it.

1. $\frac{1}{4}(3a - 5c)$

2. $\frac{2}{7}(2x - 7m)$

3. $\frac{2}{3}(5a - 2u) + \frac{1}{6}a$

4. $\frac{3}{7}(5b - 6v) - \frac{3}{4}v$

5. $\frac{1}{2}(3\frac{1}{2}x - 2\frac{1}{4}y) - 5\frac{1}{2}x$

6. $\frac{1}{3}(3\frac{1}{3}n - 2\frac{1}{4}p) + 2\frac{1}{3}p$

7. $\frac{1}{2}(3c - 5x) + \frac{1}{3}(5x - 3)$

8. $\frac{2}{3}(6u - 3b) + \frac{1}{4}(8u + 4b)$

9. $\frac{1}{7}(a + b) - \frac{1}{8}(a - b)$

10. $\frac{2}{3}(u - z) - \frac{1}{3}(u + z)$

11. $1.2(3a - 4y) + 2.3(2a + 3y)$

12. $2.7(5u - 3v) - 1.6(3v - 4u)$

13. $\frac{1}{3}(3a - 6b + 9c) + \frac{1}{2}(4a - 8b + 8c)$

14. $\frac{2}{3}(3\frac{1}{3}x + 2\frac{1}{2}y - 1\frac{1}{9}z) - \frac{1}{4}(-5\frac{1}{2}x + 4\frac{1}{3}y - 2\frac{1}{5}z)$

15. $6a(3b - 2c)$

16. $5x(2y - 7z)$

17. $2m(4k - 3b) - 7m(10k + 9b)$

18. $3ay(2a - 7y) - 12ay(y - 6a)$

19. $2(3a - 4b) - (2 - b) + (b - a) - 2(a - b)$

20. $5(x - y) - (x - y) + 3(x - y) - (y - x)$

F. Simplify.

1. $8a(3b)$
2. $6x(-2y)$
3. $(-2m)(-3n)$
4. $7xy(2xy)$
5. $-ab(-3aab)$
6. $2pq(5q)(3p)$
7. $(3a)(-2b)(-7c)$
8. $(-5x)(-3y)(2z)$
9. $4r(2rs)(-3rst)$
10. $-11xyz(-2xy)(-3yz)$
11. $\dfrac{1\frac{1}{5}c}{\frac{2}{3}}$
12. $\dfrac{\frac{3}{2}x}{\frac{5}{6}}$
13. $\dfrac{\frac{1}{2}a}{3}$
14. $(2r - 6s) \div \frac{1}{3}$
15. $(7u - 3a) \div \frac{2}{5}$
16. $\dfrac{25aabbb}{5ab}$
17. $\dfrac{16xyzz}{2xxyz}$

G. Mark 'True' or 'False'.

1. $5 \geq -5$
2. $-1.4 \leq -1.3$
3. $-9 \not> -7$
4. $7 \neq |7|$
5. $-6 \geq |-6|$
6. $7 - 3 = |3 - 7|$
7. $\dfrac{2}{73} < \dfrac{3}{110}$
8. $-.062 = \dfrac{-31}{500}$
9. $0 \not< |1 - 100|$
10. $(3 - 5)(6 - 7) = (5 - 3)(6 - 7)$
11. $(859 - 384)(7842 - 9257) = (384 - 859)(9257 - 7842)$
12. $\dfrac{8 - 2}{5 - 6} = \dfrac{2 - 8}{6 - 5}$
13. $\dfrac{58 - 72}{69 - 93} = \dfrac{58 - 72}{93 - 69}$

H. Check each of the following statements.

1. For every x and y, $(-x)(-y) \geq 0$.
2. For every x and y, $(-x)y \leq 0$.
3. For every x and y, $x + y \geq x - y$.
4. For every Δ and $\square$, $\square - \Delta = |\square - \Delta|$
5. For every $\diamond$ and x, $\diamond - x = \diamond + (-x)$

I. Complete each of the following.

1. For every a and b, the sum of 3a and 5b is _____.
2. For every y, the product of 4 and 3y is _____.
3. For every t, the quotient of 70t by 10 is _____.
4. For every x, the difference of 4x from 12 is _____.
5. For every m, if a side of square is $\frac{1}{2}$m inches long, the perimeter is _____ inches.
6. For every r and s, if a rectangle is 2r inches long and 5s inches wide, the perimeter is _____ inches.
7. For every p, r, and t, if p dollars are invested at an annual rate of r% for a period of t years, then the interest earned during the t-year period is _____ dollars.
8. For every x, y, and z, if the dimensions of a rectangular block are 2x inches, 3y inches, and 7z inches, the volume of the block is _____ cubic inches.
9. For every x, if a pile of coins contains x nickels, 3x dimes, and 2x quarters, then the pile of coins is worth _____ cents.
10. For every x, if a stack of 15x pancakes is 6 inches high then a stack of 12 such pancakes is _____ inches high.

J. Solve these problems.

1. Find the area of playground 47 yards wide and 63 yards long.
2. The cost of an article is 75% of its selling price. If the article is sold for $97.39, what is the merchant's margin?
3. The amounts of 2 ingredients in a recipe are 2 cups of sugar and 3 tablespoons of citron. This recipe will make enough dessert for 4 people. If the recipe is to be increased to take care of 8 people, how many cups

of sugar and how many tablespoons of citron will be needed?

4. A train travels from Station A to Station B in 2 hours and 20 minutes. If the distance between the two stations is 97 miles, what is the average rate of the train?

5. A baseball player's batting average at the end of a season is .302. If he was "at bat" 473 times during the season, how many hits did he get?

6. Mr. Alexander has a $15,000 insurance policy. If the annual premium rate is $11.47 per thousand dollars of insurance, what is his annual premium?

7. If Mrs. Smith buys a sofa selling at $99.50 at a discount of 18% how much does she pay for the sofa?

8. Mrs. Ashton buys a radio selling at $112. She gives $20 as a downpayment and agrees to pay $8.25 each month for a year. How much is she paying for the privilege of buying the radio on the installment plan?

9. How many hours of baby sitting at 55 cents per hour will it take to accumulate $13.75?

10. A cement sidewalk is placed around a flower bed. If the flower bed has a circumference of 45 feet and the sidewalk is 2.9 feet wide, what is the circumference of the outer edge of the sidewalk?

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